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Abstract	We consider axial-symmetric stationary flows of the ideal incompressible fluid as an important case of potential solenoid vector fields. We establish relations between axial-symmetric potential solenoid fields and principal extensions of complex analytic functions into a special topological vector space containing an infinite-dimensional commutative Banach algebra. In such a way we substantiate a method for explicit constructing axial-symmetric potentials and Stokes flow functions by means of components of the mentioned principal extensions and establish integral expressions for axial-symmetric potentials and Stokes flow functions in an arbitrary simply connected domain symmetric with respect to an axis. The obtained integral expression of Stokes flow function is applied for solving boundary problem about a streamline of the ideal incompressible fluid along an axial-symmetric body. We obtain criteria of solvability of the problem by means distributions of sources and dipoles on the axis of symmetry and construct unknown solutions using multipoles together with dipoles distributed on the axis.

Keywords (separated by '-')	Laplace Equation - Axial-symmetric potential - Stokes flow function - Streamline - Monogenic function - Analytic function

Axial-Symmetric Potential Flows

S. A. Plaksa

Dedicated to memory of my colleague and friend Igor Mel'nichenko on the occasion of his 80th birthday.

Abstract We consider axial-symmetric stationary flows of the ideal incompressible fluid as an important case of potential solenoid vector fields. We establish relations between axial-symmetric potential solenoid fields and principal extensions of complex analytic functions into a special topological vector space containing an infinite-dimensional commutative Banach algebra. In such a way we substantiate a method for explicit constructing axial-symmetric potentials and Stokes flow functions by means of components of the mentioned principal extensions and establish integral expressions for axial-symmetric potentials and Stokes flow functions in an arbitrary simply connected domain symmetric with respect to an axis. The obtained integral expression of Stokes flow function is applied for solving boundary problem about a streamline of the ideal incompressible fluid along an axial-symmetric body. We obtain criteria of solvability of the problem by means distributions of sources and dipoles on the axis of symmetry and construct unknown solutions using multipoles together with dipoles distributed on the axis.

Keywords Laplace Equation · Axial-symmetric potential · Stokes flow function Streamline · Monogenic function · Analytic function

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1 Potential Solenoid Fields and Flows

Consider a spatial stationary vector field defined by means the vector-function $\mathbf{V} \equiv \mathbf{V}(x, y, z)$ of the Cartesian coordinates x, y, z. The vector $\mathbf{V}$ is defined by means three real scalar functions $v_1 := v_1(x, y, z), v_2 := v_2(x, y, z), v_3 := v_3(x, y, z)$ which give its coordinates in the point (x, y, z), videlicet: $\mathbf{V} = (v_1, v_2, v_3)$.

Defining a *potential solenoid* field in a simply connected domain Q of the three-dimensional real space $\mathbb{R}^3$, the vector-function $\mathbf{V}$ satisfies the system of equations

$$\operatorname{div}\mathbf{V} = 0\,, \qquad \operatorname{rot}\mathbf{V} = 0\,, \tag{1}$$

where the divergence and the rotor are defined by the following equalities, respectively:

$$\operatorname{div}\mathbf{V} := \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\,,$$

$$\operatorname{rot}\mathbf{V} := \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\,, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\,, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right).$$

In particular, the velocity field of stationary flow of the ideal incompressible fluid satisfies Eq. (1) and is an important case of potential solenoid vector field.

For a potential solenoid field there exists a scalar potential function $u(x, y, z)$ such that

$$\mathbf{V} = \operatorname{grad} u := \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right),$$

and u satisfies the three-dimensional Laplace equation

$$\Delta_3 u(x, y, z) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x, y, z) = 0. \tag{2}$$

2 Plane Potential Solenoid Fields and a Complex Analytic Method of Their Description

In the case where the potential function u does not depend on the coordinate z, the field is called *plane* stationary potential solenoid field. In this case the potential function $u(x, y)$ satisfies the two-dimensional Laplace equation

$$\Delta_2 u(x, y) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x, y) = 0\,.$$

An important achievement of mathematics is the description of plane potential fields by means of analytic functions of complex variable.

A potential $u(x, y)$ and a flow function $v(x, y)$ of plane stationary potential solenoid field satisfy the Cauchy–Riemann conditions

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}, \tag{3}$$

and they form the complex potential $F(x+iy) = u(x, y) + iv(x, y)$ being an analytic function of complex variable $x + iy$. In turn, every analytic function $F(x + iy)$ satisfies the two-dimensional Laplace equation

$$\Delta_2 F(x+iy) \equiv F''(x+iy)\,(1^2+i^2) = 0$$

due to the equality $1^2 + i^2 = 0$ for the unit 1 and the imaginary unit i of the algebra of complex numbers.

Many applied problems for plane potential flows are naturally formulated in terms of flow function, and it promotes their effective solving as well as the very well advanced methods of analytic functions in the complex plane (see, e.g., Lavrentyev and Shabat 1987).

3 Axial-Symmetric Potential Solenoid Fields and Flows

In the case where a spatial potential field is symmetric with respect to the axis Ox, a potential function $u(x, y, z)$ satisfying Eq. (2) is also symmetric with respect to the axis Ox, i.e. $u(x, y, z) = \varphi(x, r) = \varphi(x, -r)$, where $r := \sqrt{y^2 + z^2}$, and φ is known as the *axial-symmetric potential*. Then in a meridian plane xOr there exists a function $\psi(x, r)$ known as the *Stokes flow function* such that the functions φ and ψ satisfy the following system of equations degenerating on the axis Ox:

$$r\,\frac{\partial \varphi(x,r)}{\partial x} = \frac{\partial \psi(x,r)}{\partial r}, \qquad r\,\frac{\partial \varphi(x,r)}{\partial r} = -\frac{\partial \psi(x,r)}{\partial x}. \tag{4}$$

Under the condition that there exist continuous second-order partial derivatives of the functions $\varphi(x, r)$ and $\psi(x, r)$, the system (4) implies the equation

$$r\Delta\varphi(x,r) + \frac{\partial \varphi(x,r)}{\partial r} = 0 \tag{5}$$

for the axial-symmetric potential and the equation

$$r\Delta\psi(x,r) - \frac{\partial \psi(x,r)}{\partial r} = 0 \tag{6}$$

for the Stokes flow function, where $\Delta := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial r^2}$.

Equations (5), (6) are particular cases of the equation

$$r\,\Delta U(x,r) + m\,\frac{\partial U(x,r)}{\partial r} = 0 \qquad (7)$$

for generalized axial-symmetric potential $U(x,r)$, where $m = \text{const} \neq 0$.

An axial-symmetric flow is one of the most widespread kinds of spatial flows. For instance, such flows are axial-symmetric flows along fuselage of aeroplanes, missiles and dirigible balloons, cumulative charges, the movement of fluids and gases in channels with round profiles etc. (cf., e.g., Kr[illegible]kov 1957; Lavrentyev and Shabat 1977, 1987; Batchelor 1970; Lavrentyev and Shabat 1987).

In view of degeneration of Eq. (4) on the axis Ox, the theory is developed considerably worse for solutions of system (4) than for solutions of system (3), i.e. complex analytic functions (see Lavrentyev and Shabat 1977, p. 18).

4 Hypercomplex Methods of Research of Spatial Potentials

Analytic function methods in the complex plane for plane potential fields inspire searching analogous effective methods for solving spatial and multidimensional problems of mathematical physics. Many such methods are based on mappings of hypercomplex algebras.

Hamilton (1866), Moisil and Theodoresco (1931), Fueter (1935), Sudbery (1979), Gürlebeck and Sprössig (1997), Kravchenko and Shapiro (1996), Leutwiler (1992), Ryan (1998), Colombo et al. (2011) and many other developed methods which are based on mappings of noncommutative algebras.

P. Ketchum (1928, 1929), Ringleb (1933), Sobrero (1934), Lorch (1943), Wagner (1948), Ward (1953), Riley (1953), Blum (1955), Roşculeţ (1955), Kunz (1971), Edenhofer (1976), Snyder (1982), I. Mel'nichenko (1975, 1986, 2003), Kovalev and Mel'nichenko (1981), Mel'nichenko and Plaksa (2008) and many other developed methods which are based on mappings of commutative algebras.

Last decades, the hypercomplex analysis in both commutative and nonconnutative algebras is very intensively developing. Its applications are developed for constructing solutions of equations of mathematical physics (especially, the multidimensional Laplace equation (see, e.g., Plaksa and Shpakovskii 2011; Plaksa 2012; Plaksa and Shpakivskyi 2012, 2014; 2017; Plaksa and Pukhtaievych 2013, 2014; Shpakivskyi 2016), the Helmholtz equation (see, e.g., Kravchenko and Shapiro 1996), the Klein–Gordon equation (see, e.g., Kravchenko 2009), the Navier–Stokes equation (see, e.g., Cerejeiras and Kähler 2000; Binlin Zhang et al. 2014; Gürlebeck and Habetha 2016; Grigor'ev 2017), the biharmonic equation (see, e.g., Gryshchuk and Plaksa 2009 2013, 2016, 2017), the equations for axial-symmetric potential and for generalized axial-symmetric potential, the equation for Stokes flow function and other elliptic equations degenerating on an axis (see, e.g., Mel'nichenko and Plaksa 1996, 1997, 2004, 2008; Plaksa 2009, 2012, 2013). In fact, studying analytic functions of a complex variable, hyperholomorphic and monogenic functions defined in commutative and noncommutative algebras discovers a way to develop effective analytic methods for solving various problems of mathematical physics.

In particular, we proved in the papers Mel'nichenko and Plaksa (1997, 2008) that solutions of the system (4) in a domain convex in the direction of the axis Or are constructed by means components of principal extensions of analytic functions of a complex variable into a corresponding domain of a special two-dimensional vector manifold in an infinite-dimensional commutative Banach algebra.

5 An Infinite-Dimensional Commutative Banach Algebra $\mathbb{H}_{\mathbb{C}}$ and a Topological Vector Space $\widetilde{\mathbb{H}}_{\mathbb{C}}$ Containing the Algebra $\mathbb{H}_{\mathbb{C}}$

Let $\mathbb{H}_{\mathbb{C}} := \{a = \sum\limits_{k=1}^{\infty} a_k e_k \ : \ a_k \in \mathbb{C}, \sum\limits_{k=1}^{\infty} |a_k| < \infty\}$ be a commutative associative Banach algebra over the field of complex numbers $\mathbb{C}$ with the norm $\|a\|_{\mathbb{H}_{\mathbb{C}}} := \sum\limits_{k=1}^{\infty} |a_k|$ and the following multiplication table for elements of the basis $\{e_k\}_{k=1}^{\infty}$:

$$e_n e_1 = e_n, \quad e_m e_n = \frac{1}{2}\left(e_{m+n-1} + (-1)^{n-1} e_{m-n+1}\right) \quad \forall\ m \geq n \geq 1$$

(cf., e.g., Mel'nichenko and Plaksa 1997; 2008). The multiplication table was offered by Mel'nichenko 1984.

The algebra $\mathbb{H}_{\mathbb{C}}$ is isomorphic to the algebra $\mathbf{F}_{\mathbf{cos}}$ of absolutely convergent trigonometric Fourier series

$$c(\tau) = \sum_{k=1}^{\infty} c_k\, i^{k-1} \cos(k-1)\tau$$

with complex coefficients c_k and the norm $\|c\|_{\mathbf{F}_{\mathbf{cos}}} := \sum\limits_{k=1}^{\infty} |c_k|$. In this case, we have the isomorphism $e_{2k-1} \longleftrightarrow i^{k-1} \cos(k-1)\tau$ between basic elements.

Consider the Cartesian plane $\mu := \{\zeta = xe_1 + re_2 \ : \ x, r \in \mathbb{R}\}$ which is a linear span of the elements e_1, e_2 over the field of real numbers $\mathbb{R}$.

For a domain $D \subset \mathbb{R}^2$ we use consentaneous denotations for congruent domains of the plane μ and the complex plane $\mathbb{C}$, namely: $D_\zeta := \{\zeta = xe_1 + re_2 \ : \ (x, r) \in D\} \subset \mu$ and $D_z := \{z = x + ir \ : \ (x, r) \in D\} \subset \mathbb{C}$.

We proved in the papers Mel'nichenko and Plaksa (1997, 2008) that in the case where the domain D is convex in the direction of the axis Or, solutions of the system

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(4) can be constructed by means components of principal extensions of complex functions analytic in D_z into the domain D_ζ.

To generalize such a relation between solutions of the system (4) and hypercomplex functions for domains of more general form, let us insert the algebra $\mathbb{H}_{\mathbb{C}}$ in the topological vector space $\widetilde{\mathbb{H}}_{\mathbb{C}} := \{g = \sum_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{C}\}$ with the topology of coordinate-wise convergence.

Note that $\widetilde{\mathbb{H}}_{\mathbb{C}}$ is not an algebra because the product of elements $g_1, g_2 \in \widetilde{\mathbb{H}}_{\mathbb{C}}$ is defined not always. But for each $g = \sum_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{H}}_{\mathbb{C}}$ and $\tilde{\zeta} = (x + iy)e_1 + re_2$, where $x, y, r \in \mathbb{R}$, one can define the product

$$g\tilde{\zeta} \equiv \tilde{\zeta} g := \left(c_1(x+iy) - \frac{c_2}{2}\, r\right)e_1 + \left(c_2(x+iy) + \left(c_1 - \frac{c_3}{2}\right) r\right)e_2 +$$

$$+ \sum_{k=3}^{\infty} \left(c_k(x+iy) + \frac{1}{2}\left(c_{k-1} - c_{k+1}\right) r\right)e_k \,.$$

6 Monogenic Functions Taking Values in the Space $\widetilde{\mathbb{H}}_{\mathbb{C}}$

Below, we shall consider functions given in domains of the plane μ and taking values in the space $\widetilde{\mathbb{H}}_{\mathbb{C}}$.

We say that a continuous function $\Phi : Q_\zeta \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ is *monogenic* in a domain $Q_\zeta \subset \mu$ if Φ is differentiable in the sense of Gateaux in every point of Q_ζ, i.e. if for every $\zeta \in Q_\zeta$ there exists an element $\Phi'(\zeta) \in \widetilde{\mathbb{H}}_{\mathbb{C}}$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\, \Phi'(\zeta) \quad \forall\, h \in \mu \,. \tag{8}$$

Let us note that we use the notion of monogenic function in the sense of existence of derived numbers for this function in the domain Q_ζ (cf. Goursat 1910; Trokhimchuk 1964).

Consider the decomposition

$$\Phi(\zeta) = \sum_{k=1}^{\infty} U_k(x, r)\, e_k \,, \qquad \zeta = xe_1 + re_2 \,, \tag{9}$$

of a function $\Phi : Q_\zeta \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ with respect to the basis $\{e_k\}_{k=1}^{\infty}$.

Below, we suppose that the functions $U_k : Q \to \mathbb{C}$ are $\mathbb{R}$-differentiable in the domain Q, i.e.

$$U_k(x+\Delta x, r+\Delta r) - U_k(x,r) = \frac{\partial U_k(x,r)}{\partial x}\,\Delta x + \frac{\partial U_k(x,r)}{\partial r}\,\Delta r +$$

$$+\,o\left(\sqrt{(\Delta x)^2+(\Delta r)^2}\right), \qquad (\Delta x)^2+(\Delta r)^2 \to 0\,,$$

for all $(x,r)\in Q$. Evidently, such an assumption implies the fact that the function (9) is continuous in Q_{ζ}.

In the following theorem we establish the necessary and sufficient conditions for a function $\Phi : \Omega_\zeta \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ be monogenic in a domain $\Omega_\zeta \subset \mu$.

Theorem 1 *Let in the decomposition (9) of a function $\Phi : Q_\zeta \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ the functions $U_k : Q \to \mathbb{C}$ be $\mathbb{R}$-differentiable in Q. In order the function Φ be monogenic in the domain Q_ζ, it is necessary and sufficient that the following Cauchy–Riemann condition be satisfied in Q_ζ:*

$$\frac{\partial \Phi}{\partial r} = \frac{\partial \Phi}{\partial x}\, e_2\,. \tag{10}$$

Proof *Necessity*. If the function $\Phi : Q_\zeta \to \widetilde{\mathbb{H}}_{\mathbb{C}}$ is monogenic in the domain Q_ζ, then for $h = e_1$ the equality (8) turns into the equality

$$\Phi'(\zeta) = \frac{\partial \Phi(\zeta)}{\partial x}\,.$$

Now, setting this expression and $h = e_2$ in the equality (8), we obtain the condition (10).

Sufficiency. Let us write the conditions (10) in expanded form:

$$\begin{aligned}
\frac{\partial U_1(x,r)}{\partial r} &= -\frac{1}{2}\,\frac{\partial U_2(x,r)}{\partial x}\,,\\
\frac{\partial U_2(x,r)}{\partial r} &= \frac{\partial U_1(x,r)}{\partial x} - \frac{1}{2}\,\frac{\partial U_3(x,r)}{\partial x}\,,\\
\frac{\partial U_k(x,r)}{\partial r} &= \frac{1}{2}\,\frac{\partial U_{k-1}(x,r)}{\partial x} - \frac{1}{2}\,\frac{\partial U_{k+1}(x,r)}{\partial x}\,, \qquad k = 3, 4, \ldots\,,
\end{aligned} \tag{11}$$

Let $\varepsilon > 0$ and $h := h_1e_1 + h_2e_2$, where $h_1, h_2 \in \mathbb{R}$. Taking into account the equalities (11), we have

$$\big(\Phi(\zeta+\varepsilon h) - \Phi(\zeta)\big)\,\varepsilon^{-1} - h\,\Phi'(\zeta) =$$

$$= \left(\sum_{k=1}^{\infty}\Big(U_k(x+\varepsilon h_1, r+\varepsilon h_2) - U_k(x,r)\Big)e_k -\right.$$

$$\left. -\,\varepsilon\,(h_1e_1+h_2e_2)\sum_{k=1}^{\infty}\frac{\partial U_k(x,r)}{\partial x}\,e_k\right)\varepsilon^{-1} =$$

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$$= \sum_{k=1}^{\infty} \varepsilon^{-1} \Bigg(U_k(x+\varepsilon h_1, r+\varepsilon h_2) - U_k(x,r) - \\ - \frac{\partial U_k(x,r)}{\partial x}\, \varepsilon h_1 - \frac{\partial U_k(x,r)}{\partial r}\, \varepsilon h_2 \Bigg) e_k \,. \qquad (12)$$

Inasmuch as the functions U_k are $\mathbb{R}$-differentiable in Q, the last series converges coordinate-wise to zero, i.e. the function Φ is monogenic in Q_ζ. Theorem is proved.

7 Principal Extensions of Complex Analytic Functions and Its Relations to Axial-Symmetric Potential Fields

Let us construct for every complex analytic function a special monogenic function taking values in the space $\widetilde{\mathbb{H}}_{\mathbb{C}}$. Such a monogenic function is a generalization of the principal extension of complex analytic function into a commutative Banach algebra. We establish below relations between generalized principal extensions of complex analytic functions and axial-symmetric potential solenoid fields. In such a way we substantiate a method for explicit constructing axial-symmetric potentials and Stokes flow functions in an arbitrary simply connected domain symmetric with respect to the axis Ox by means of components of the mentioned principal extensions.

Let the domain $D \subset \mathbb{R}^2$ be symmetric with respect to the axis Ox and the domain D_z be simply connected. Let the boundary ∂D_z of domain D_z cross the real axis at the points b_1 and b_2. We assume that $b_1 < b_2$.

For every $z \in D_z \setminus \mathbb{R}$ let us fix an arbitrary Jordan rectifiable curve $\Gamma_{z\bar{z}}$ in the domain D_z that is symmetric with respect to the real axis $\mathbb{R}$ and connects the points z and $\bar{z}$. In addition, we shall agree that in the case where the domain D_z is unbounded, the curve $\Gamma_{z\bar{z}}$ crosses the real axis $\mathbb{R}$ on the interval $(-\infty, b_1)$.

For $z \in D_z \setminus \mathbb{R}$ let $\sqrt{(t-z)(t-\bar{z})}$ be that continuous branch of the analytic function $G(t) = \sqrt{(t-z)(t-\bar{z})}$ outside of the cut along $\Gamma_{z\bar{z}}$ for which $G(b_2) > 0$.

For every $z \in D_z$ with $\mathrm{Im}\, z = 0$, we define by continuity $\sqrt{(t-z)(t-\bar{z})} := t - z$ for $z < b_2$, and $\sqrt{(t-z)(t-\bar{z})} := -(t-z)$ for $z > b_2$.

For every function $F : D_z \to \mathbb{C}$ analytic in a domain D_z consider the function

$$\frac{1}{2\pi i} \int\limits_{\gamma} (te_1 - \zeta)^{-1} F(t)\, dt = U_1(x,r)\, e_1 + 2 \sum_{k=2}^{\infty} U_k(x,r)\, e_{2k-1} \qquad (13)$$

given in D_ζ and taking values in $\widetilde{\mathbb{H}}_{\mathbb{C}}$, where

$$U_k(x,r) := \frac{1}{2\pi i} \int\limits_{\gamma} \frac{F(t)}{\sqrt{(t-z)(t-\bar{z})}} \left(\frac{\sqrt{(t-z)(t-\bar{z})} - (t-x)}{r} \right)^{k-1} dt, \qquad (14)$$

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$\zeta = xe_1 + re_3$ and $z = x + ir$ for $(x, r) \in D$, and γ is an arbitrary closed Jordan rectifiable curve in D_z which embraces $\Gamma_{z\bar{z}}$.

Let us note that if to take a domain $D' \subset D$ which is symmetric with respect to the axis Ox and convex in the direction of the axis Or, and to fix the segment connecting the points z and $\bar{z}$ as the curve $\Gamma_{z\bar{z}}$ for every $z \in D'_z \setminus \mathbb{R}$, then the function (13) turns into the principal extension of the analytic function F into the domain D'_ζ (see Hille and Phillips 1957, p. 165).

Therefore, we shall consider the function (13) as a *principal extension* of analytic function $F : D_z \to \mathbb{C}$ into the domain D_ζ.

The following theorem generalizes Theorem 2.6 in Mel'nichenko and Plaksa (2008) (cf. also Theorem 18 in Mel'nichenko and Plaksa 1997), which describes relations between principal extensions of analytic functions into the plane μ and solutions of the system (4) in domains convex in the direction of the axis Or.

Theorem 2 *Let the domain $D \subset \mathbb{R}^2$ be symmetric with respect to the axis Ox and the domain D_z be simply connected. If $F : D_z \to \mathbb{C}$ is an analytic function in a domain D_z, then the first and the second components of the function (13) generate the solutions φ and ψ of system (4) in D by the formulas*

$$\varphi(x, r) = U_1(x, r), \quad \psi(x, r) = r\, U_2(x, r)\,. \tag{15}$$

Moreover, the functions φ and ψ defined by the formulas (15) are solutions in D of Eqs. (5) and (6), respectively.

Proof In view of the equality (14) and Cauchy theorem, the functions (15) have the form

$$\varphi(x, r) = \frac{1}{2\pi i} \int_\gamma \frac{F(t)}{\sqrt{(t - z)(t - \bar{z})}}\, dt, \tag{16}$$

$$\psi(x, r) = -\frac{1}{2\pi i} \int_\gamma \frac{F(t)\,(t - x)}{\sqrt{(t - z)(t - \bar{z})}}\, dt\,. \tag{17}$$

Now, substituting the partial derivatives of functions (16) and (17) into the equations of system (4) and Eqs. (5) and (6), one can see that the mentioned equations become identities in the domain D. Theorem is proved.

Thus, the formulas (15) enable to construct axial-symmetric potentials and Stokes flow functions by means of components of principal extensions of complex analytic functions into the plane μ.

In particular, elementary functions of the variable $\zeta = xe_1 + re_2$ are principal extensions of corresponding elementary functions of a complex variable. Let us write expansions with respect to the basis $\{e_k\}_{k=1}^{\infty}$ of some elementary functions of the variable $\zeta = xe_1 + re_2$. Note that in view of isomorphism between the algebras $\mathbb{H}_{\mathbb{C}}$ and $\mathbf{F_{cos}}$, the construction of expansions of this sort is reduced to the determination of relevant Fourier coefficients.

The expansion of a power function has the form (see Mel'nichenko and Plaksa 1997; 2008)

$$\zeta^n = (x^2 + r^2)^{n/2}\left(P_n(\cos\vartheta)\, e_1 + 2\sum_{k=1}^{n} \frac{(\operatorname{sgn} r)^k\, n!}{(n+k)!}\, P_n^k(\cos\vartheta)\, e_{k+1}\right),$$

where n is a positive integer, $\cos\vartheta := x(x^2+r^2)^{-1/2}$,

$$\operatorname{sgn} r := \begin{cases} 1 \text{ for } r \geq 0, \\ -1 \text{ for } r < 0, \end{cases}$$

and Legendre polynomials P_n and associated Legendre polynomials P_n^m are defined be the equalities

$$P_n(t) := \frac{1}{2^n\, n!}\,\frac{d^n}{dt^n}\,(t^2 - 1)^n, \qquad P_n^m(t) := (1-t^2)^{m/2}\,\frac{d^m}{dt^m}\, P_n(t)\,.$$

For the functions e^{ζ}, $\sin\zeta$ and $\cos\zeta$ we have

$$e^{\zeta} = e^x\left(J_0(r)\, e_1 + 2\sum_{k=1}^{\infty} J_k(r)\, e_{k+1}\right),$$

$$\sin\zeta = \sin x\left(J_0(ir)\, e_1 + 2\sum_{k=1}^{\infty} J_{2k}(ir)\, e_{2k+1}\right) - 2i\,\cos x \sum_{k=1}^{\infty} J_{2k-1}(ir)\, e_{2k},$$

$$\cos\zeta = \cos x\left(J_0(ir)\, e_1 + 2\sum_{k=1}^{\infty} J_{2k}(ir)\, e_{2k+1}\right) + 2i\,\sin x \sum_{k=1}^{\infty} J_{2k-1}(ir)\, e_{2k},$$

where Bessel functions J_m are defined by the equality

$$J_m(t) := \frac{(-1)^m}{\pi}\int_0^{\pi} e^{it\cos\tau}\,\cos m\tau\, d\tau\,.$$

In the following theorem we describe relations between components U_k of hypercomplex monogenic function (13) and solutions of elliptic equations degenerating on the axis Ox.

Theorem 3 *Let the domain $D \subset \mathbb{R}^2$ be symmetric with respect to the axis Ox and the domain D_z be simply connected. If $F : D_z \to \mathbb{C}$ is an analytic function in a domain D_z, then the components U_k of principal extension (13) of function F into the domain D_ζ satisfy the equations*

$$r^2\,\Delta U_k(x,r) + r\,\frac{\partial U_k(x,r)}{\partial r} - (k-1)^2\,U_k(x,r) = 0, \quad k = 1, 2, \ldots,$$

in the domain D. In addition, the function

$$\psi_k(x,r) := r^{k-1}\,U_k(x,r)$$

is a solution in D of the equation

$$r\;\Delta\psi_k(x,r) - (2k-3)\,\frac{\partial\psi_k(x,r)}{\partial r} = 0\,, \quad k = 1, 2, \ldots\,.$$

Theorem 2 follows from the equalities (13), (14) and Theorem 3.1 in Plaksa (2009).

8 Integral Expressions for Axial-Symmetric Potentials and Stokes Flow Functions in Boundary Value Problems

Boundary value problems for solutions of elliptic equations have numerous applications in mathematical physics. For the two- and three-dimensional Laplace equations, various methods for the efficient solving of boundary value problems are developed. However, the direct application of these methods to solving of boundary value problems for axial-symmetric potentials and Stokes flow functions is a quite complicated problem due to a degeneration of the Eqs. (5), (6) on the axis Ox.

In the paper Keldysh (1951), some correct statements of boundary value problems for an elliptic equation with a degeneration on a straight line are described. They have shown certain differences of these problems from boundary value problems for elliptic equations without degeneration.

Therefore, for solving of boundary value problems in a meridian plane of an axial-symmetric potential field, it is necessary to develop special methods that take into account the nature and specific features of axial-symmetric problems.

One of ways for researching axial-symmetric problems is based on representation of its solutions in the form of potentials of a simple or double layer. With this purpose, fundamental solutions of the appropriate equations with partial derivatives are used (cf., e.g., Weinstein 1948, 1953, 1962; Mikhailov and Rajabov 1972; Rajabov 1974). For instance, in such a way in the paper Rajabov (1974), the main boundary value problems for solutions of equation (7) in a domain with the Lyapunov boundary was reduced to the Fredholm integral equations.

Many methods of research of elliptic equations are based on integral expressions of solutions via analytic functions of a complex variable (cf. Whittaker and Watson 1927; Bateman 1944; Henrici 1953,1957; Huber 1954; Mackie 1955; Erdelyi 1956; Gilbert 1969; Krivenkov 1957; 1960; Rajabov 1965,1968; Polozhii 1973; Polozhii and Ulitko 1965; Kapshivyi 1972; Aleksandrov et al. 1979; Mel'nichenko 1984).

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8.1 Integral Expressions for Axial-Symmetric Potentials and Stokes Flow Functions

The formulas (16) and (17) generate axial-symmetric potentials and Stokes flow functions in an arbitrary simply connected domain symmetric with respect to the axis Ox.

Below, we formulate four statements on the representability of axial-symmetric potentials and Stokes flow functions by the formulas (16) and (17), respectively. The cases of a bounded domain D and an unbounded domain D are considered separately.

In the case of a bounded domain D, the following two statements are true.

Theorem 4 (Plaksa 2001, 2003; Mel'nichenko and Plaksa 2008) *Suppose that a function $\varphi(x, r)$ is even with respect to the variable r and satisfies Eq. (5) in a bounded domain D symmetric with respect to the axis Ox. In this case, there exists the unique function F analytic in the domain D_z and satisfying the condition*

$$F(\bar{z}) = \overline{F(z)} \qquad \forall\, z \in D_z \tag{18}$$

and such that the equality (16) is fulfilled for all $(x, r) \in D$.

Theorem 5 (Plaksa 2003; Mel'nichenko and Plaksa 2008) *Suppose that the function $\psi(x, y)$ is even with respect to the variable r and satisfies Eq. (6) in a bounded domain D symmetric with respect to the axis Ox and the additional assumption*

$$\psi(x, 0) \equiv 0 \quad \forall\, (x, 0) \in D. \tag{19}$$

In this case, there exists a function F_0 analytic in the domain D_z such that the equality (17) is fulfilled with $F = F_0$ for all $(x, r) \in D$. Moreover, any analytic function F which satisfies the condition (18) and the equality (17) for all $(x, r) \in D$ is expressed in the form $F(z) = F_0(z) + C$, where C is a real constant.

The requirement (19) is natural. For example, for the model of steady flow of an ideal incompressible fluid without sources and vortexes it means that the axis Ox is a line of flow.

In the case of a bounded domain D with the bounded Jordan boundary, the following two statements are true.

Theorem 6 (Plaksa 2002; Mel'nichenko and Plaksa 2008) *Suppose that a function $\varphi(x, r)$ satisfies Eq. (5) in an unbounded domain D with the bounded Jordan boundary symmetric with respect to the axis Ox. Suppose also that the function $\varphi(x, r)$ is even with respect to the variable r and is vanishing at infinity. In this case, there exists the unique analytic in D_z function F vanishing at infinity and satisfying the condition (18) and such that the equality (16) is fulfilled for all $(x, r) \in D$.*

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Theorem 7 (~~Mel'nichenko and~~ Plaksa 2003, Mel'nichenko and Plaksa 2008) *Suppose that a function $\psi(x, r)$ satisfies Eq. (6) and the condition (19) in an unbounded domain D with the bounded Jordan boundary symmetric with respect to the axis Ox. Suppose also that the function $\varphi(x, r)$ is even with respect to the variable r and is vanishing at infinity. In this case, there exists the unique analytic in D_z function F vanishing at infinity and satisfying the condition (18) and such that the equality (17) is fulfilled for all $(x, r) \in D$. Moreover, the function F has a zero at least of the second order at infinity.*

It follows from Theorem 2 that the functions (16), (17) satisfies the system (4) in the domain D for every function F analytic in D_z. But these functions takes real values if and only if the condition (18) is satisfied.

Thus, all axial-symmetric potentials and Stokes flow functions, i.e. solutions of the system (4) in D with a physical interpretation, are represented by the integral expressions (16), (17) which can be used for solving boundary value problems for axial-symmetric potential solenoid fields.

Let us note that if the boundary ∂D_z is a Jordan rectifiable curve and the function F belongs to the Smirnov class E_1 (see, e.g., Privalov 1950, p. 205) in the domain D_z, then the formulas (16) and (17) can be transformed to the form

$$\varphi(x, r) = \frac{1}{2\pi i} \int\limits_{\partial D_z} \frac{F(t)}{\sqrt{(t - z)(t - \bar{z})}}\, dt\,, \tag{20}$$

$$\psi(x, r) = -\frac{1}{2\pi i} \int\limits_{\partial D_z} \frac{F(t)\,(t - x)}{\sqrt{(t - z)(t - \bar{z})}}\, dt \tag{21}$$

for all $(x, r) \in D$, where $F(t)$ are the angular boundary values of the function F which, as it is known (see, e.g., Privalov 1950, p. 205), exist at almost all points $t \in \partial D_z$. In the case where D is an unbounded domain, we admit additionally that the function F is vanishing at infinity for obtaining the formula (20) or the function F has a zero at least of the second order at infinity for obtaining the formula (21).

In the papers Plaksa (2001), Mel'nichenko and Plaksa (2008) we established sufficient conditions for continuous continuations of the functions (20), (21) on the boundary ∂D of a domain D and obtained estimations for modules of continuity of boundary values of the mentioned functions.

Using the integral expressions (20), (21) of axial-symmetric potentials and Stockes flow functions, we developed a functional analytic method for effective solving boundary problems in a meridian plane of spatial axial-symmetric potential field (see Plaksa 2001, 2002, 2003; Mel'nichenko and Plaksa 2008).

8.2 Integral Equation for an Outer Dirichlet Boundary Value Problem for the Stokes Flow Function

Below, let D be an unbounded domain in the meridian plane xOr and the boundary ∂D is a closed Jordan rectifiable curve symmetric with respect to the axis Ox. The closure of domain D is denoted by $\overline{D}$.

Let us consider the following *outer Dirichlet boundary value problem for the Stokes flow function*: to find a continuous in $\overline{D}$ function $\psi(x,r)$ which is a solution of Eq. (6) in D when its boundary values $\psi_{\partial D}(x,r)$ are given on the boundary ∂D, i.e. $\psi(x,r) = \psi_{\partial D}(x,r)$ for all $(x,r) \in \partial D$. It is also required that the function ψ is vanishing at infinity and satisfies the identity (19).

Note that, vanishing at infinity and satisfying the identity (19), the Stokes flow function $\psi(x,r)$ satisfies the maximum principle in the domain D. It follows from the maximum principle that a solution of the mentioned Dirichlet problem is unique.

Let us remind that the boundary ∂D_z of domain D_z cross the real axis at the points b_1, b_2 and $b_1 < b_2$.

For every $z \in \partial D_z \setminus \mathbb{R}$, by $\Gamma_{z\bar{z}}$ we denote that Jordan subarc of the boundary ∂D_z with the end points z and $\bar{z}$ which contains the point b_1. For $z \in \partial D_z \setminus \mathbb{R}$ let $\sqrt{(t-z)(t-\bar{z})}$ be that continuous branch of the analytic function $G(t) = \sqrt{(t-z)(t-\bar{z})}$ outside of the cut along $\Gamma_{z\bar{z}}$ for which $G(b_2) > 0$.

The direction of the circuit of ∂D_z with the domain D_z to the left is taken to be the positive direction.

If the function F has the properties stipulated in Theorem 7, then we shall call F the *creative function* for the function $\psi(x,r)$.

It is established in the papers Plaksa (2003), Mel'nichenko and Plaksa (2008) that the solution of the mentioned Dirichlet problem is expressed in the form (21), where the creative for ψ function F is a solution of the integral equation

$$-\frac{1}{2\pi i}\int\limits_{\partial D_z} \frac{F(t)(t-x)}{\sqrt{(t-z)(t-\bar{z})}}\,dt = \psi_{\partial D}(x,r), \quad \forall\,(x,r) \in \partial D\,. \tag{22}$$

Here values of the function $\sqrt{(t-z)(t-\bar{z})}$ for $t \in \Gamma_{z\bar{z}}$ are taken on the right side of the cut $\Gamma_{z\bar{z}}$.

In the papers Plaksa (2003), Mel'nichenko and Plaksa (2008) we developed a method for a transition of Eq. (22) to the Cauchy singular integral equation on the real axis.

In a case important for applications where ∂D_z is a smooth curve satisfying certain additional requirements, then the mentioned singular integral equation is reduced to the Fredholm integral equation of the second kind. Moreover, it is established in Plaksa (2003), Mel'nichenko and Plaksa (2008) that in this case there exists the unique function F which satisfies Eq. (22) and is creative for the solution $\psi(x,r)$ of the mentioned Dirichlet problem.

Let us note that we obtained the Fredholm integral equation for the Dirichlet boundary value problem for the Stokes flow function in the case where the boundary ∂D_z belongs to a class being wider than the class of Lyapunov curves.

Let us note else that in the case where ∂D_z is a circle, the solution F of Eq. (22) is obtained explicitly (see Plaksa 2003; Mel'nichenko and Plaksa 2008).

9 Boundary Value Problem About a Steady Streamline of the Ideal Incompressible Fluid Along an Axial-Symmetric Body

Consider an outer boundary problem having important applications in the hydrodynamics of potential flows.

Let us consider the following *problem about a steady streamline* of the ideal incompressible fluid along an axial-symmetric body: to find a solution $\psi_1(x,r)$ of Eq. (6) in D that satisfies the condition

$$\psi_1(x,r) = 0 \quad \forall\ (x,r) \in \partial D \cup \{(x,r) \in D\ :\ r = 0\} \tag{23}$$

and have the following asymptotic

$$\psi_1(x,r) = \frac{1}{2}\, v_\infty r^2 + o(1), \quad x^2 + r^2 \to \infty,\ \ v_\infty > 0. \tag{24}$$

For the model of steady flow of the ideal incompressible fluid the condition (23) means that the boundary ∂D and the axis Ox are lines of flow. In the asymptotic (24) v_∞ is a velocity of unbounded flow at infinity.

We note that explicit solutions of such a problem are known in certain particular cases of steady streamline along an axial-symmetric body (see Lavrentyev and Shabat 1977; Loitsyanskii 1987; Batchelor 1970; Weinstein 1953; Mel'nichenko and Pik 1973, 1975).

Inasmuch as the Stokes flow function

$$\psi(x,r) = \psi_1(x,r) - \frac{1}{2}\, v_\infty r^2$$

is vanishing at infinity, we can apply the integral expression (21) for solving the boundary value problem about a steady streamline of the ideal incompressible fluid along an axial-symmetric body.

To solve this problem in such a way, we obtain the integral equation

$$\frac{1}{\pi i} \int\limits_{\partial D_z} \frac{F(t)(t-x)}{\sqrt{(t-z)(t-\bar{z})}}\, dt = v_\infty r^2, \quad (x,r) \in \partial D : r \neq 0, \tag{25}$$

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where it is necessary to find the function F creative for $\psi(x, r)$. Evidently, Eq. (25) is a particular case of Eq. (22).

In addition, using the integral expression (21) for the Stokes flow function, in the papers Mel'nichenko and Plaksa (2003, 2008) we obtained some results having a natural physical interpretation. Namely, for a boundary problem about a streamline of the ideal incompressible fluid along an axial-symmetric body, we obtained criteria of solvability by means distributions of sources and dipoles on the axis of symmetry and constructed unknown solutions using multipoles together with dipoles distributed on the axis.

9.1 Expressions of Solutions via Distributions of Sources on the Axis

Consider a source located on the axis Ox at the point $(x_0, 0)$ with the intensity q. Such a source is simulated by means of analytic function $F(t) = q/(t - x_0)$ for which the following flow function corresponds by the formula (17):

$$\psi(x, r) = -q\, \frac{x - x_0}{\sqrt{(x - x_0)^2 + r^2}}\,.$$

As a result of an interaction between an flow of the ideal incompressible fluid oncoming with the velocity $v_\infty > 0$ and a source with the intensity $q > 0$ located at the point $(x_1, 0)$ and a source with the intensity $-q$ (i.e. a sink) located at the point $(x_2, 0)$ in the case $x_1 < x_2$ one can obtain the solution

$$\psi_1(x, r) = \frac{1}{2}\, v_\infty r^2 - q\, \frac{x - x_1}{\sqrt{(x - x_1)^2 + r^2}} + q\, \frac{x - x_2}{\sqrt{(x - x_1 2^2 + r^2}} \tag{26}$$

of problem about a steady streamline of the ideal incompressible fluid along an axial-symmetric oval body, for which the boundary points satisfy the equality $\psi_1(x, r) = 0$. Lines of flow are given by the equations $\psi_1(x, r) = \text{const}$ (see, e.g., Lavrentyev and Shabat 1977, p. 201).

In the case where $x_1 > x_2$, the function (26) is no solution of problem about a steady streamline because, evidently, there are exist no points satisfying the equality $\psi_1(x, r) = 0$ for any $x \in (x_2, x_1)$.

If sources with intensities q_k are located on the axis Ox in points $(x_k, 0)$, respectively, then in order the function

$$\psi_1(x, r) = \frac{1}{2}\, v_\infty r^2 - \sum_{k=1}^{n} q_k\, \frac{x - x_k}{\sqrt{(x - x_k)^2 + r^2}}$$

be a solution of the problem about a steady streamline, it is necessary (but it is not sufficient, generally speaking) that the total intensity of sources be

$$\sum_{k=1}^{n} q_k = 0\,. \tag{27}$$

Now, suppose that the function $F(z)$ is analytic in $\mathbb{C} \setminus [a_1, a_2]$, where $[a_1, a_2]$ is a segment of the real axis. Let $F^+(t)$ or $F^-(t)$ denote its boundary values on $[a_1, a_2]$ when $z \to t$ from a half-plane upper or lower with respect to the real axis, respectively. Denote by $L_p[a_1, a_2]$ the set of functions summable on $[a_1, a_2]$ to the p th power.

Using the Cauchy theorem, it is easy to prove the following theorem having a natural physical interpretation.

Theorem 8 *Suppose that the solution F of Eq. (25) has the form*

$$F(z) = \sum_{k=1}^{n} \frac{q_k}{z - x_{1,k}} + F_1(z)\,, \tag{28}$$

where all $x_{1,k}$ belong to a segment $[a_1, a_2]$, the equality (27) is fulfilled and the function F_1 can be continued to an analytic function outside of the segment $[a_1, a_2]$ and its boundary values $F_1^+(t)$, $F_1^-(t)$ belong to $L_p[a_1, a_2]$, $p > 1$. Then the solution of the problem about steady streamline is given by the formula

$$\psi_1(x, r) = \frac{v_\infty r^2}{2} - \sum_{k=1}^{n} q_k \frac{x - x_{1,k}}{\sqrt{(x - x_{1,k})^2 + r^2}} +$$

$$+ \int_{a_1}^{a_2} \frac{q(t)(t - x)}{\sqrt{(t - x)^2 + r^2}}\, dt \quad \forall\ (x, r) \in D\,, \tag{29}$$

where

$$q(t) := -\frac{1}{2\pi i}\,(F_1^+(t) - F_1^-(t)) \equiv -\frac{1}{\pi}\,\mathrm{Im}\, F_1^+(t)$$

is the distribution density of intensity of sources on $[a_1, a_2]$ which correspond to the function F_1. Moreover, the total intensity of such sources is

$$\int_{a_1}^{a_2} q(t)\, dt = 0. \tag{30}$$

Theorem 8 generalizes the corresponding theorem in Mel'nichenko and Plaksa (2003, 2008), where the case $q_k \equiv 0$ was considered. Let us note that the formula

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(29) with $q_k \equiv 0$ is well-known classical result (see Lavrentyev and Shabat 1977, p. 201). At the same time, in this case, Theorem 8 enables to find the distribution density of sources intensity via boundary values of the creative function F_1 on the set of sources distribution.

It is easy to prove the following theorem converse to Theorem 8.

Theorem 9 *Suppose that the solution of the problem about steady streamline is given by the formula (29), where* $q(t) \in L_p[a_1, a_2]$, $p > 1$, *and the equalities (27), (30) are fulfilled. Then the solution* F *of Eq. (25) has the form (28), where the function* F_1 *can be continued to the function analytic outside of the segment* $[a_1, a_2]$ *and its boundary values* $F_1^+(t)$, $F_1^-(t)$ *belong to* $L_p[a_1, a_2]$. *Moreover, in this case*

$$F_1(z) = -\int\limits_{a_1}^{a_2} \frac{q(t)}{t-z}\, dt \qquad \forall\ z \in \mathbb{C} \setminus [a_1, a_2]. \tag{31}$$

Theorem 9 generalizes the corresponding theorem in Mel'nichenko and Plaksa (2003, 2008), where the case $q_k \equiv 0$ was considered.

It is possible to rewrite the formula (29) in a more short form if to introduce a Lebesgue–Stieltjes measure generated by the following function of bounded variation:

$$\mu(t) := \int\limits_{a_1}^{t} q(\tau)\, d\tau + \sum_{k=1}^{n} q_k\, \theta(t - x_{1,k})\,, \tag{32}$$

where

$$\theta(\tau) := \begin{cases} 1 \text{ for } \tau \geq 0, \\ 0 \text{ for } \tau < 0 \end{cases}$$

is the Heaviside function. Then the formula (29) can be rewritten as

$$\psi_1(x, r) = \frac{v_\infty r^2}{2} + \int\limits_{a_1}^{a_2} \frac{(t-x)}{\sqrt{(t-x)^2 + r^2}}\, d\mu(t) \quad \forall\ (x, r) \in D\,.$$

We can use as well the following formal equality:

$$d\mu(t) = \left(q(t) + \sum_{k=1}^{n} q_k\, \delta(t - x_{1,k}) \right) dt\,,$$

where δ is the Dirac delta function.

Thus, for all $(x, r) \in D$, the formula (29) can be also rewritten as

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$$\psi_1(x,r) = \frac{v_\infty r^2}{2} + \int_{a_1}^{a_2} \left(q(t) + \sum_{k=1}^{n} q_k \delta(t - x_{1,k}) \right) \frac{(t-x)}{\sqrt{(t-x)^2 + r^2}}\, dt\,.$$

Let's agree that integrals on unlimited intervals of the real axis are understood in the sense of the principal value.

The following theorem was essentially proved in Mel'nichenko and Plaksa (2003, 2008).

Theorem 10 *The distribution density $q(t)$ of intensity of sources, which correspond to the function F_1, is expressed via values of the function (31) on the set $(-\infty, b_1) \cup (b_2, \infty)$ in the form of the repeated integral*

$$q(t) = \frac{b_2 - b_1}{2\pi^2 \sqrt{(b_2 - t)(t - b_1)}} \int_{-\infty}^{\infty} A(t,\xi) \int_{-\infty}^{\infty} B(\xi,\tau)\, d\tau\, d\xi \quad \forall\, t \in [b_1, b_2] \tag{33}$$

in the case when it exists. Here

$$A(t,\xi) := \mathrm{ch}\,(\pi\xi) \exp\left(i\xi \ln \frac{t - b_1}{b_2 - t} \right),$$

$$B(\xi,\tau) := -\frac{F(b_1 + (b_2 - b_1)(\mathrm{cth}\,\frac{\tau}{2} + 1)/2)}{\exp(\tau) - 1} \exp(-i\tau\xi).$$

Proof Consider the integral equation (31), in which $a_1 = b_1, a_2 = b_2$ and $z \in (-\infty, b_1) \cup (b_2, \infty)$. Using the change of variables $t = b_1 + (b_2 - b_1)\frac{\tau}{\tau+1}$ and $z = b_1 + (b_2 - b_1)\frac{\xi}{\xi - 1}$, we transform it to the integral equation

$$\int_0^\infty \frac{q_*(\tau)}{\tau + \xi}\, d\tau = F_*(\xi), \qquad \xi > 0, \tag{34}$$

in which

$$F_*(\xi) := -F\left(b_1 + (b_2 - b_1)\frac{\xi}{\xi - 1} \right) / (\xi - 1)$$

and

$$q_*(\tau) := q\left(b_1 + (b_2 - b_1)\frac{\tau}{\tau + 1} \right) / (\tau + 1).$$

Equation (34) is solvable explicitly (see, e.g., Zabreiko and al 1968, p. 30). As a result of the inversion of integral operator in Eq. (34), we obtain the expression (33) for the distribution density $q(t)$ of intensity of sources on the segment $[b_1, b_2]$. The theorem is proved.

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9.2 Expressions of Solutions via Distributions of Dipoles on the Axis

Consider a dipole located on the axis Ox at the point $(x_0, 0)$. Let the moment p of dipole be directed along the axis Ox. We shall also call p by the intensity of dipole. Such a dipole is simulated by means of analytic function $F(t) = p/(t - x_0)^2$ for which the following flow function corresponds by the formula (17):

$$\psi(x, r) = p\,\frac{r^2}{((x - x_0)^2 + r^2)^{3/2}}\,.$$

It is well known that as a result of an interaction between an flow of the ideal incompressible fluid oncoming with the velocity $v_\infty > 0$ and a dipole with the intensity $-p$ located at the point $(0, 0)$, one can obtain the picture of steady streamline along a ball with the radius $R = \sqrt[3]{\frac{2p}{v_\infty}}$ and the center in the origin (see, e.g., Lavrentyev and Shabat 1977, p. 200) Lines of flow are given by the equations

$$\psi_1(x, r) = \frac{v_\infty r^2}{2} - p\,\frac{r^2}{(x^2 + r^2)^{3/2}} = \text{const.}$$

Let us note that inasmuch as the solution F of Eq. (25) has a zero at least of the second order at infinity, then F has a primitive function $\mathcal{F}$ in D which is vanishing at infinity.

Now, using the Cauchy theorem, it is easy to prove the following theorem having also a natural physical interpretation.

Theorem 11 *Suppose that the solution F of Eq. (25) has the form*

$$F(z) = \sum_{k=1}^{m} \frac{p_k}{(z - x_{2,k})^2} + F_2(z)\,, \tag{35}$$

where all $x_{2,k}$ belong to a segment $[a_1, a_2]$ and a primitive function $\mathcal{F}_2$ for the function F_2 can be continued to an analytic function outside of the segment $[a_1, a_2]$ and its boundary values $\mathcal{F}_2^+(t)$, $\mathcal{F}_2^-(t)$ belong to $L_p[a_1, a_2]$, $p > 1$. Then the solution of the problem about steady streamline is given by the formula

$$\psi_1(x, r) = \frac{v_\infty r^2}{2} + \sum_{k=1}^{m} p_k\,\frac{r^2}{((x - x_{2,k})^2 + r^2)^{3/2}} - \\ - r^2 \int_{a_1}^{a_2} \frac{p(t)}{((t - x)^2 + r^2)^{3/2}}\,dt \quad \forall\ (x, r) \in D\,, \tag{36}$$

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where

$$p(t) := -\frac{1}{2\pi i}\,(\mathcal{F}_2^+(t) - \mathcal{F}_2^-(t)) \equiv -\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{F}_2^+(t)$$

is the distribution density of intensity of dipoles on $[a_1, a_2]$ *which correspond to the function* $\mathcal{F}_2$.

It is also easy to prove the following theorem converse to Theorem 11.

Theorem 12 *Suppose that the solution of the problem about steady streamline is given by the formula (36), where* $p(t) \in L_p[a_1, a_2]$, $p > 1$. *Then the solution* F *of Eq. (25) has the form (35), where the function* F_2 *has a primitive function* $\mathcal{F}_2$ *that can be continued to an function analytic outside of the segment* $[a_1, a_2]$ *and its boundary values* $\mathcal{F}_2^+(t)$, $\mathcal{F}_2^-(t)$ *belong to* $L_p[a_1, a_2]$. *Moreover, in this case*

$$\mathcal{F}_2(z) = -\int_{a_1}^{a_2} \frac{p(t)}{t - z}\,dt \quad \forall\, z \in \mathbb{C} \setminus [a_1, a_2]. \tag{37}$$

The following theorem is proved similarly to Theorem 10.

Theorem 13 *The distribution density* $p(t)$ *of intensity of dipoles, which correspond to the function* $\mathcal{F}_2$, *is expressed via values of the function (37) on the set* $(-\infty, b_1) \cup (b_2, \infty)$ *in the form of the repeated integral*

$$p(t) = \frac{b_2 - b_1}{2\pi^2\sqrt{(b_2 - t)(t - b_1)}} \int_{-\infty}^{\infty} A(t, \xi) \int_{-\infty}^{\infty} C(\xi, \tau)\,d\tau\,d\xi \quad \forall\, t \in [b_1, b_2]$$

in the case when it exists. Here the function $A(t, \xi)$ *is defined in Theorem 10 and*

$$C(\xi, \tau) := -\frac{\mathcal{F}_2(b_1 + (b_2 - b_1)(\mathrm{cth}\,\frac{\tau}{2} + 1)/2)}{\exp(\tau) - 1}\,\exp(-i\tau\xi)\,.$$

Theorem 11–13 generalize the corresponding theorem in Mel'nichenko and Plaksa (2003, 2008), where the case $p_k \equiv 0$ was considered.

Let us note that for a source with the intensity $q > 0$ located at the point $(x_1, 0)$ and a source with the intensity $-q$ located at the point $(x_2, 0)$ in the case $x_1 < x_2$, the following equality holds:

$$-q\,\frac{x - x_1}{\sqrt{(x - x_1)^2 + r^2}} + q\,\frac{x - x_2}{\sqrt{(x - x_2)^2 + r^2}} =$$

$$= -r^2 \int_{a_1}^{a_2} \frac{q}{((t - x)^2 + r^2)^{3/2}}\,dt \quad \forall\,(x, r) \in D\,.$$

Therefore, such a pair of sources can be replaced by dipoles located on the segment $[x_1, x_2]$ with the distribution density of their intensity $p(t) = q$ for all $t \in [x_1, x_2]$.

Taking into account this note, it is easy to conclude that every solution of the problem about steady streamline of the form (29) is expressed also by the formula (36), where $p_k \equiv 0$, $p(t) = \mu(t)$ and the function $\mu(t)$ is defined by the equality (32).

But among domains D for which the solution of the problem about steady streamline is expressed by the formula (36), there are domains for which the function ψ_1 can not be expressed as (29). For example, the last statement is true in the case where there exists $p_k \neq 0$ in the equality (36). Thus, the class of domains for which the solution of the problem about steady streamline is given by the formula (36) is wider than the class of domains for which the solution of the mentioned problem is given by the formula (29).

If to introduce the function

$$\widetilde{p}(t) := p(t) - \sum_{k=1}^{m} p_k\, \delta(t - x_{2,k})$$

and a Lebesgue–Stieltjes measure generated by the following function of bounded variation:

$$\nu(t) := \int_{a_1}^{t} p(\tau)\, d\tau - \sum_{k=1}^{m} p_k\, \theta(t - x_{2,k})\,,$$

then the formula (36) can be rewritten as

$$\psi_1(x, r) = \frac{v_\infty r^2}{2} - r^2 \int_{a_1}^{a_2} \frac{d\nu(t)}{((t-x)^2 + r^2)^{3/2}} =$$

$$= \frac{v_\infty r^2}{2} - r^2 \int_{a_1}^{a_2} \frac{\widetilde{p}(t)}{((t-x)^2 + r^2)^{3/2}}\, dt \quad \forall\ (x, r) \in D\,. \tag{38}$$

Let us note that if the function (38) is the solution of the problem about steady streamline and, in addition, $\widetilde{p}(t) \geq 0$ for all $t \in [a_1, a_2]$, then the axial-symmetric body $\mathbb{R}^2 \setminus D$ is convex in the direction of the axis Or. It follows evidently from a monotonicity with respect to r^2 of the integral in the formula (38).

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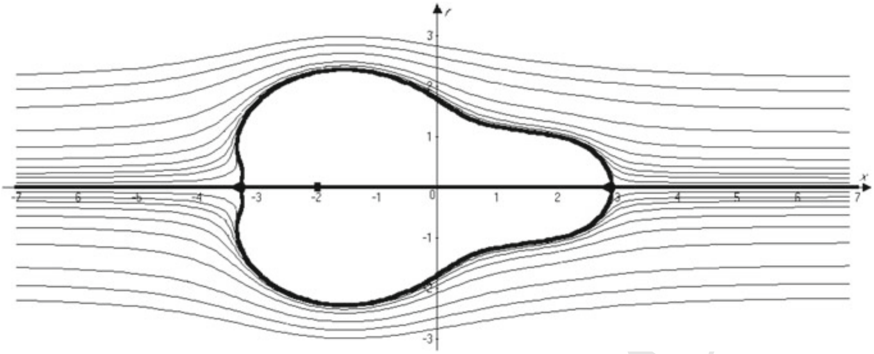

Fig. 1 The streamline along a "pear"

9.3 *Using Multipoles for a Construction of Solutions of the Problem About Steady Streamline*

At the same time, there are also domains for which it is necessary to use multipoles together with dipoles to obtain the solution of the problem about steady streamline.

For example, the streamline along a "pear" is represented on Fig. 1. In this case, lines of flow are given by the equations

$$\psi_1(x,r) = r^2 \left(1,7 - \frac{44(x+2)}{((x+2)^2+r^2)^{5/2}} - \frac{20}{((x+2,245)^2+r^2)^{3/2}} - \right.$$

$$\left. - \frac{1}{((x-1)^2+r^2)^{3/2}} - \frac{1}{((x-2)^2+r^2)^{3/2}} \right) = \text{const}.$$

The solution ψ_1 is obtained by means of three dipoles located in the points $(-2,245;0)$, $(1;0)$, $(2;0)$ and a quadrupole located in the point $(-2;0)$. To construct this solution, we use the formula (17) in which the flow function for a quadrupole

$$\psi(x,r) = -\frac{44(x+2)r^2}{((x+2)^2+r^2)^{5/2}}$$

corresponds to the creative function $F(t) = -\frac{88}{3(t+2)^3}$. Let us note that in this case a "pear" $\mathbb{R}^2 \setminus D$ is not convex in the direction of the axis Or.

The streamline along a "matreshka" is represented on Fig. 2. In this case, it turns out already that the body $\mathbb{R}^2 \setminus D$ is convex in the direction of the axis Or. Lines of flow are given by the equations

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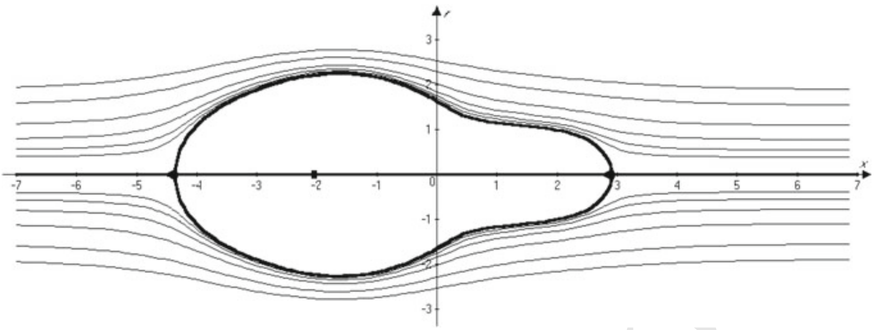

Fig. 2 The streamline along a "matreshka"

$$\psi_1(x,r) = r^2\left(1,7 - \frac{44(x+2)}{((x+2)^2+r^2)^{5/2}} - \frac{20}{((x+2,5)^2+r^2)^{3/2}} -\right.$$

$$\left. - \frac{1}{((x-1)^2+r^2)^{3/2}} - \frac{1}{((x-2)^2+r^2)^{3/2}}\right) = \text{const.}$$

The change of the streamline picture is obtained due to a displacement of a dipole from the point $(-2,245;0)$ into the point $(-2,5;0)$.

An essential specificity for applications is a fact that an use of multipoles can give no solution of the problem about steady streamline. A combination of dipoles and multipoles gives a streamline picture only if certain relations between their intensities are fulfilled.

9.4 Interaction Between a Flow and a Pair "Dipole and Quadrupole"

Let us consider an interaction between a flow of the ideal incompressible fluid oncoming with the velocity $v_\infty > 0$ and a dipole and a quadrupole, which are located on the axis Ox.

Let a quadrupole of intensity m be located at the point $(0;0)$ and a dipole of intensity p be located at the point $(x_0;0)$, $x_0 \neq 0$.

We consider two cases: $x_0 > 0$ (see Fig. 3) and $x_0 < 0$ (see. Fig. 4).

(1) In the case where $x_0 > 0$ we use the function

$$\psi(x,r) = -m\frac{3xr^2}{(x^2+r^2)^{5/2}} - p\frac{r^2}{((x-x_0)^2+r^2)^{3/2}} \tag{39}$$

that corresponds to the creative function $F(t) = -\frac{2m}{t^3} - \frac{p}{(t-x_0)^2}$ in accordance with the formula (17).

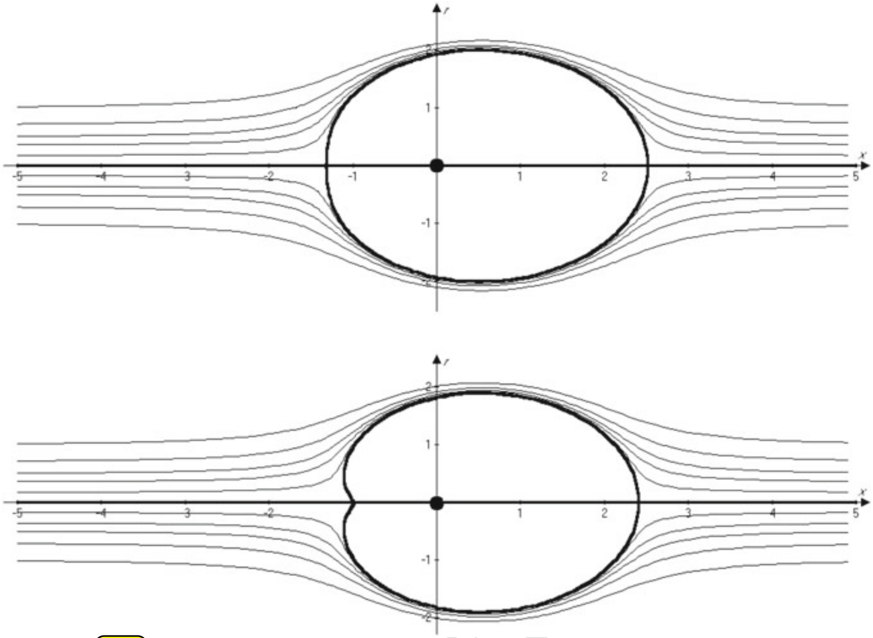

Fig. 3 A Dipole is located on the right of a quadrupole

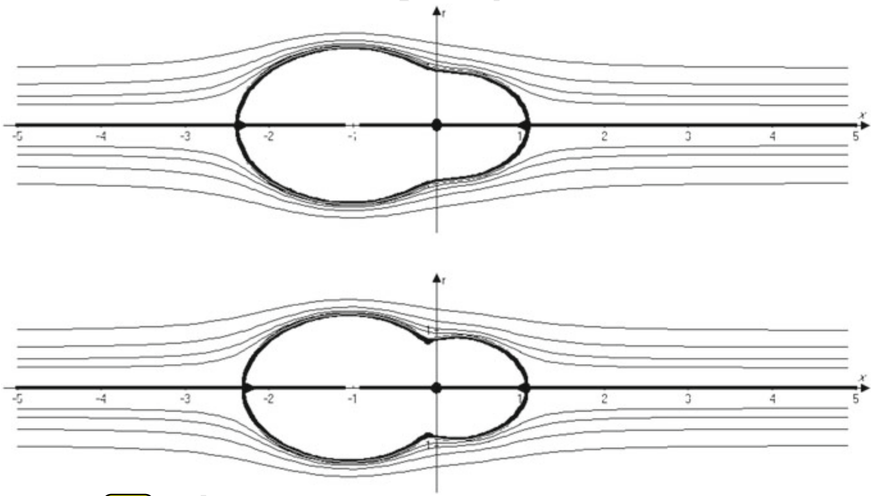

Fig. 4 A Dipole is located on the left of a quadrupole

If the intensity p of the dipole is small in comparison with the intensity m of the quadrupole (the quantitative relation is formulated below), then the singularity (0, 0) of the function $\psi(x, r)$ is located on the boundary that has the equation

$$\frac{v_\infty}{2} + \frac{\psi(x,r)}{r^2} = 0$$

which, taking into account the equality (39), we rewrite as

$$\frac{v_\infty}{2} - \frac{3xm}{(x^2+r^2)^{5/2}} - \frac{p}{((x-x_0)^2+r^2)^{3/2}} = 0. \tag{40}$$

Therefore, the function

$$\psi_1(x,r) = \frac{v_\infty r^2}{2} - m\frac{3xr^2}{(x^2+r^2)^{5/2}} - p\frac{r^2}{((x-x_0)^2+r^2)^{3/2}} \tag{41}$$

is no solution of the problem about steady streamline.

An increase of the dipole intensity results finally in formation of a closed contour Γ for which coordinates of points satisfy the Eq. (40) and, moreover, the point $(0,0)$ is located inside of the domain bounded by Γ. Then the equality (40) is fulfilled at a point $(x,0)$ with $x<0$, i.e. we have the following equality:

$$\frac{v_\infty}{2} + \frac{3m}{x^4} - \frac{p}{(x_0-x)^3} = 0\,,$$

from which we find

$$p = \frac{v_\infty}{2}(x_0-x)^3\left(1+\frac{6m}{v_\infty x^4}\right). \tag{42}$$

The equality (42) is fulfilled at a point $(x,0)$ with $x<0$ if and only if

$$p \ge \frac{v_\infty}{2}\min_{x<0}\left((x_0-x)^3\left(1+\frac{6m}{v_\infty\, x^4}\right)\right) =: c_1\,,$$

and in this case the function (41) gives the solution of the problem about steady streamline for unbounded domain with the boundary Γ.

Now, we can assert:

(a) if $p \ge c_1$, *then the function* (41) *is the solution of the problem about steady streamline for a certain domain* D;

(b) if $p < c_1$, *then there is no domain* D, *for which the function* (41) *would be the solution of the problem about steady streamline.*

One can see the examples of streamline picture in the cases $p > c_1$ (the upper picture on Fig. 3) and $p = c_1$ (the lower picture on Fig. 3).

(2) In the case where $x_0 < 0$ we conclude that if the point $(0,0)$ is located inside of the domain bounded by the contour Γ, then for each $x \in [x_0, 0]$ there exists a point (x,r) with $r>0$ that the equality (40) is fulfilled, from which we find

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$$p=\frac{v_\infty}{2}((x-x_0)^2+r^2)^{3/2}\left(1-\frac{6mx}{v_\infty(x^2+r^2)^{5/2}}\right). \tag{43}$$

For each fixed $x\in[x_0,0]$, the equality (43) is fulfilled at a point (x,r) with $r>0$ if and only if

$$p\geq\frac{v_\infty}{2}\min_{r\geq0}\left((x-x_0)^2+r^2)^{3/2}\left(1-\frac{6mx}{v_\infty(x^2+r^2)^{5/2}}\right)\right).$$

Finally, for each $x\in[x_0,0]$ there exists a point (x,r) with $r>0$ in which the equality (43) is fulfilled if and only if

$$p\geq\frac{v_\infty}{2}\max_{x\in[x_0,0]}\min_{r\geq0}\left(((x-x_0)^2+r^2)^{3/2}\left(1-\frac{6mx}{v_\infty\,(x^2+r^2)^{5/2}}\right)\right)=:c_2\,.$$

Thus, the following statements are true:

(a) if $p\geq c_2$ *,then the function* (41) *is the solution of the problem about steady streamline for a certain domain* D;

(b) if $p<c_2$ *, then there is no domain* D*, for which the function* (41) *would be the solution of the problem about steady streamline.*

One can see the examples of streamline picture in the cases $p>c_2$ (the upper picture on Fig. 4) and $p=c_2$ (the lower picture on Fig. 4).

Let us note that the boundary ∂D is piecewise-smooth if $p=c_1$ in the case where $x_0>0$ (see the lower picture on Fig. 3) or $p=c_2$ in the case where $x_0<0$ (see the lower picture on Fig. 4). AQ1

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