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QUATERNION-VALUED MEASURE AND ITS TOTAL VARIATION

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The notion of a measure is one of the most fundamental objects in mathematics and it would be superfluous to talk much about this. We present now a few lines only in order to explain what we are going to do in the paper, for more details the reader is referred, for instance, to [1], but for many other sources as well.

Let X be a non-empty set and let $\mathfrak{M}$ be a σ-algebra of subsets of X. A measure (sometimes called a positive measure) is a function μ defined on the measurable space $(X, \mathfrak{M})$ whose range is in $[0, \infty] =: \overline{\mathbb{R}}_+$ and which is countably additive, i.e., if $\{A_i\}$ is a disjoint countable family of elements of $\mathfrak{M}$ then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i). \tag{1}$$

This definition includes tacitly that the series on the right-hand side converges to a non-negative number or to ∞.

We assume that there exists at least one $A \in \mathfrak{M}$ for which $\mu(A) < \infty$. This excludes the trivial situation of the measure identically equal to ∞.

Some important properties are:

1. $\mu(\emptyset) = 0$.

2. Any measure is finite additive, i.e., holds for a finite number of pair-wise disjoint elements of $\mathfrak{M}$.

3. Any measure is monotone: if A, B are in $\mathfrak{M}$ and $A \subset B$ then $\mu(A) \leq \mu(B)$.

4. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$, $A = \cup_{n=1}^{\infty} A_n$, $A_1 \subset A_2 \subset \cdots \subset A_n, \ldots$, then $\mu(A_n) \longrightarrow \mu(A)$ as $n \longrightarrow \infty$.

5. If $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$, $A_1 \supset A_2 \supset \cdots \supset A_n \ldots$, $A = \cap_{n=1}^{\infty} A_n$, $\mu(A_1) < \infty$, then $\mu(A_n) \longrightarrow \mu(A)$ as $n \longrightarrow \infty$.

Definition 1. *A measure on a measurable space $(X, \mathfrak{M})$ is called σ-finite if there exists a collection of sets $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ such that $\cup_{n=1}^{\infty} A_n = X$ and for each $n \geq 1$ $\mu(A_n) < \infty$.*

Let us recall a notion of a signed measure or charge.

Definition 2. *A signed measure (or a charge) on a measurable space $(X, \mathfrak{M})$ is a function*

$$\lambda: \mathfrak{M} \to \mathbb{R} \cup \{-\infty, \infty\} \tag{2}$$

such that $\lambda(\emptyset) = 0$ and λ is countably additive.

The origin of the notion of the measure explains why it takes just non-negative values. At the same time the question arises: can the measure be complex-valued?

A *complex measure w* is a complex-valued countably additive function defined on

$\mathfrak{M}$. A good source of basic information may be Chapter 6 of the book [2].

In accordance with the definition if w is identically zero then w is a positive measure. A positive measure is allowed to have $+\infty$ as its value; but it is proved that a complex measure μ has as its values the complex numbers only: any $\mu(E)$ is in $\mathbb{C}$. The *real measures* are defined as σ-additive real-valued functions and they form a subclass of the complex measures. Complex measures are not monotone in general but they verify the other above properties. It is worth noting that for a given σ-algebra the collections of positive and of complex measures have, in general, a non-empty intersection but the former is not necessarily a subcollection of the latter; the same kind of relation exists between the positive and the real measures.

The definition of a complex measure can be rephrased as follows. Consider a countable family $\{E_i\}$ of elements of $\mathfrak{M}$ which are pairwise disjoint and let $E := \bigcup_{i=1}^{\infty} E_i$; the family $\{E_i\}$ is called a partition of E. Then a complex measure w is a complex function on $\mathfrak{M}$ such that

$$w(E) = \sum_{i=1}^{\infty} w(E_i) \tag{3}$$

for any $E \in \mathfrak{M}$ and for every partition $\{E_i\}$ of E.

Notice that the requirement of being $\{E_i\}$ in (3) any partition of E has a strong implication: one can change the order of the enumeration in $\{E_i\}$, thus every rearrangement of the series is convergent to the same complex number; it is known that hence the series in (3) converges in fact absolutely.

The main goal of this work is to show that some ideas from [2] extend onto σ-additive functions with values in Hamilton quaternions [3].

We assume in the sequel that X is a non-empty set.

Definition 3. *Let $\mathfrak{M}$ be a σ-algebra of subsets of a set X. A quaternionic measure ω on a measurable space $(X, \mathfrak{M})$ is a quaternion-valued function on $\mathfrak{M}$ such that for any collection of sets $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ that $A_n \cap A_m = \emptyset$ whenever $n \neq m$ we have*

$$\omega\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \omega(A_n). \tag{4}$$

Since the union of sets A_n is not changed if the subscripts are permuted, every rearrangement of series (4) must converge to $\omega(\bigcup_{n=1}^{\infty} A_n)$. For this reason, we assume that the series converges absolutely.

Let us ask the question: Is it possible to find a positive measure μ on a measurable space $(X, \mathfrak{M})$ such that $|\omega(A)| \leq \mu(A)$ for any $A \in \mathfrak{M}$? That is, we ask to find a positive measure μ that dominates the Euclidean module of ω. It is easily seen that if there exists such a dominant measure then for any partition $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$, we have:

$$\sum_{n=1}^{\infty} |\omega(A_n)| \leq \sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Let us define the set function $var[\omega](\cdot)$ on $\mathfrak{M}$ as follows:

$$var[\omega](A) := \sup \sum_{n=1}^{\infty} |\omega(A_n)|,$$

where the supremum is taken over all partitions of A. It is clear that

$$|\omega(A)| \leq var[\omega](A) \leq \mu(A).$$

We will call the function $var[\omega]$ *the total variation* of ω.

Theorem 1. *The total variation $var[\omega]$ of a quaternionic measure ω on a measurable space $(X, \mathfrak{M})$ is a positive measure on $(X, \mathfrak{M})$.*

Proof. Suppose $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ is a partition of A. Let $\{A_{nm}\}$ be a partition of $A_n, n \in \mathbb{N}$. Hence, we have:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\omega(A_{nm})| \leq var[\omega](A).$$

Then, taking into account that $A_n = \bigcup_{m=1}^{\infty} A_{nm}$, we have:

$$\sum_{n=1}^{\infty} \sup \sum_{m=1}^{\infty} |\omega(A_{nm})| \leq var[\omega](A).$$

Hence,

$$\sum_{n=1}^{\infty} var[\omega](A_n) \leq var[\omega](A). \tag{5}$$

Let us show that

$$\sum_{n=1}^{\infty} var[\omega](A_n) \geq var[\omega](A).$$

Suppose $\{B_m\}$ is a partition of A. Then for a fixed $m \in \mathbb{N}$, the collection $\{B_m \cap A_n\}_{n\in\mathbb{N}}$ is a partition of B_m and for a fixed $n \in \mathbb{N}$, the collection $\{B_m \cap A_n\}_{m\in\mathbb{N}}$ is a partition of A_n. Thus, we have:

$$\sum_{m=1}^{\infty} |\omega(B_m)| = \sum_{m=1}^{\infty} \left|\sum_{n=1}^{\infty} \omega(B_m \cap A_n)\right| \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\omega(B_m \cap A_n)|$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\omega(B_m \cap A_n)| \leq \sum_{n=1}^{\infty} |\omega(A_n)|. \tag{6}$$

Since Eq. (6) holds for every partition $\{B_m\}$ of A, it holds that

$$var[\omega](A) \leq \sum_{n=1}^{\infty} |\omega(A_n)|.$$

Therefore, together with (5) one obtains:

$$var[\omega](A) = \sum_{n=1}^{\infty} var[\omega](A_n).$$

It is easily seen that

$$var[\omega](\emptyset) = 0. \qquad \blacksquare$$

Some comments on this Theorem are given in [4].

Theorem 2. *If ω is a quaternionic measure on a measurable space $(X, \mathfrak{M})$, then*

$$var[\omega](X) < \infty.$$

Proof. First of all we need an auxiliary inequality.

Suppose $h_1, \dots, h_n$ are arbitrary quaternions, then there exists a subset S of $\{1, \dots, n\}$ such that

$$\left|\sum_{l\in S} h_l\right| \geq \frac{3(\pi^2 - 8)}{4\pi^3} \sum_{l=1}^{n} |h_l|. \tag{7}$$

Every quaternion $q = q_0 + \vec{q}$, where q_0 is the scalar part and $\vec{q}$ the vector part of q, can be represented in the following form

$$q = \frac{q_0}{|q|} + \frac{\vec{q}}{|\vec{q}|}\frac{|\vec{q}|}{|q|} = |q|\left(\cos\alpha + \frac{\vec{q}}{|\vec{q}|}\sin\alpha\right),$$

where α is a solution of the system of equations $\cos\alpha = \frac{q_0}{|q|}$ and $\sin\alpha = \frac{|\vec{q}|}{|q|}$. It is easily seen that this system has a unique solution α_0 in the segment $0 \leq \alpha \leq \pi$. One can show that there is a unique vector $\vec{v}_0$ such that $\vec{v}_0$ and $\vec{q}$ have same direction and $|\vec{v}_0| = \alpha_0$.

Thus, every quaternion has the following unique representation

$$q = |q|\left(\cos|\vec{v}_0| + \frac{\vec{v}_0}{|\vec{v}_0|}\sin|\vec{v}_0|\right), 0 \leq |\vec{v}_0| \leq \pi. \tag{8}$$

Write $h_l = |h_l|\left(\cos|\vec{v}_l| + \frac{\vec{v}_l}{|\vec{v}_l|}\sin|\vec{v}_l|\right)$, where $\vec{v}_l = \alpha_l I + \beta_l J + \gamma_l K$, $0 \leq |\vec{v}_l| \leq \pi$, is vector as $\vec{v}_0$ in Eq 8.

Consider $\vec{\theta} = \theta_1 I + \theta_2 J + \theta_3 K$, where $0 \le \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2} \le \pi$ and let $S(\vec{\theta})$ be a set of all $l \in S$ such that $\cos(|\vec{v}_l - \vec{\theta}|) > 0$. Then

$$\left|\sum_{l\in S(\vec{\theta})} h_l\right| = \left|\sum_{l\in S(\vec{\theta})} h_l e^{-\vec{\theta}}\right| \ge \mathrm{Re} \sum_{l\in S(\vec{\theta})} h_l e^{-\vec{\theta}} = \sum_{l=1}^{n} |h_l| \cos^+(|\vec{v}_l - \vec{\theta}|),$$

where $\cos^+(|\vec{v}_l - \vec{\theta}|) = \cos(|\vec{v}_l - \vec{\theta}|)\ I_{\{\cos(|\vec{v}_l - \vec{\theta}|) > 0\}}$.

Choose $\vec{\theta}_0$ so as to maximize last sum, and put $S(\vec{\theta}_0)$. This maximum is at least as large as the average of the sum over $\vec{\theta} = \theta_1 I + \theta_2 J + \theta_3 K$, and this average is $\frac{3(\pi^2-8)}{4\pi^3} \sum_{l=1}^{n} |h_l|$, because

$$\frac{1}{m(B(\pi))} \iiint_{|\vec{v}_l - \vec{\theta}| \le \pi} \cos^+(|\vec{v}_l - \vec{\theta}|)\ d\vec{\theta} =$$
$$\frac{1}{m(B(\pi))} \iiint_{|\vec{\theta}| \le \pi} \cos^+(|\vec{\theta}|)\ d\overrightarrow{\theta =}$$
$$\frac{1}{m(B(\pi))} \iiint_{|\vec{\theta}| \le \frac{\pi}{2}} \cos(|\vec{\theta}|)\ d\vec{\theta} =$$
$$\frac{3}{4\pi^4} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(\theta)\ \cos(\pi\rho)\ \rho^2 d\rho d\theta d\varphi = \frac{3(\pi^2-8)}{4\pi^3},$$

where $m(B(\pi)) = \frac{4}{3}\pi^4$ is the volume of the ball of radius π.

We now proceed to prove the inequality (7).

Suppose that there is a set $A \in \mathfrak{M}$ such that $var[w](A) = \infty$. Put $t = \frac{4\pi^3}{3(\pi^2-8)}(1 + |w(A)|)$. Since $var[w](A) > t$ there is a partition $\{A_i\}$ of A such that

$$\sum_{i=1}^{n} |w(A_i)| > t$$

for some n. Let us apply Lemma with $h_i = w(A_i)$ to conclude that there is a set $E \subset A$ which is a union of some sets A_i and

$$|w(E)| > \frac{3(\pi^2-8)}{4\pi^3} t > 1.$$

Considering $F = A \backslash E$, it follows that

$$|w(F)| = |w(A) - w(E)| \ge |w(E)| - |w(A)| > \frac{3(\pi^2-8)}{4\pi^3} t - |w(A)| = 1.$$

Thus, we have split A into disjoint sets E and F such that $|w(E)| > 1$ and $|w(F)| > 1$.

Now, if $var[w](X) = \infty$ then we can split X into sets E_1 and F_1 with $|w(E_1)| > 1$ and $var[w](F_1) = \infty$. Then we split F_1 into E_2 and F_2 with $|w(E_2)| > 1$ and $var[w](F_2) = \infty$. Continuing in this way, we obtain countably infinite disjoint collection $\{E_n\}$ with $|w(E_n)| > 1$ for all n. The countable additivity of w implies that

$$w(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} w(E_n).$$

But this series cannot converge since $w(E_n)$ does not tend to 0 as $n \to \infty$. This contradiction shows that $var[w](X) < \infty$. ■

Remark 1. *The common term measure includes $+\infty$ as an admissible value. Thus the measures do not form a subclass of the quaternionic measures.*

A detailed justification of these results can be found in the paper [5].

References:

1. Halmos, P. R. (1950). Measure Theory : Springer-Verlag New York, Heidelberg, Berlin.

2. Rudin, W. (1987). Real and Complex Analysis : McGraw-Hill Book Company, Singapore, New York.

3. Hamilton, W. R. (1866). Elements of quaternions : University of Dublin Press.

4. Agrawal, S., Kulkarni, S. H. (2000). An analogue of the Riesz-representation theorem. *Novi Sad J. Math*. Vol. 30. P. 143–154.

5. Luna-Elizarrarás, M. E., Pogorui, A., Shapiro, M., Kolomiiets, T. (2020). On Quaternionic Measure. *Advances in Applied Clifford Algebras*. Vol. 30, iss. 4, art. 63. P. 1–17.