On compact classes of solutions of Dirichlet problem in simply connected domains

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Let D be a domain in $\mathbb{C}$. In what follows, a mapping $f: D \to \mathbb{C}$ is assumed to be *sense-preserving,* moreover, we assume that f has partial derivatives almost everywhere. Put $f_{\overline{z}} = (f_x + if_y)/2$ and $f_z = (f_x - if_y)/2$. The *complex dilatation* of f at $z \in D$ is defined as follows: $\mu(z) = \mu_f(z) = f_{\overline{z}}/f_z$ for $f_z \neq 0$ and $\mu(z) = 0$ otherwise. The *maximal dilatation* of f at z is the following function: $K_\mu(z) = K_{\mu_f}(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$.

Consider the following Cauchy problem:

$f_{\overline{z}} = \mu(z) \cdot f_z\,,$

$\lim\limits_{\zeta \to P} \mathrm{Re}\, f(\zeta) = \varphi(P) \qquad \forall\ P \in E_D\,,$

where $\varphi: E_D \to \mathbb{R}$ is a predefined continuous function. In what follows, we assume that D is some simply connected domain in $\mathbb{C}$. The solution of this problem is called *regular,* if one of two conditions is fulfilled: or $f(z) = const$ in D, or f is an open discrete $W_{\mathrm{loc}}^{1,1}(D)$-mapping such that $J(z, f) \neq 0$ for almost any $z \in D$.

Given $z_0 \in D$, a function $\varphi: E_D \to \mathbb{R}$, a function $\Phi: \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ and a function $\mathcal{M}(\Omega)$ of open sets $\Omega \subset D$, we denote by $\mathfrak{F}_{\varphi,\Phi,z_0}^{\mathcal{M}}(D)$ the class of all regular solutions $f: D \to \mathbb{C}$ of the Cauchy problem that satisfy the condition $\mathrm{Im}\, f(z_0) = 0$ and, in addition, $\int\limits_{\Omega} \Phi(K_\mu(z)) \cdot \frac{dm(z)}{(1+|z|^2)^2} \leqslant \mathcal{M}(\Omega)$

for any open set $\Omega \subset D$.

Theorem. *Let D be some simply connected domain in $\mathbb{C}$, and let $\Phi: \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ be a continuous increasing convex function which satisfies the condition* $\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$

for some $\delta > \Phi(0)$. Assume that the function $\mathcal{M}$ is bounded, and the function φ in Cauchy problem is continuous. Then the family $\mathfrak{F}_{\varphi,\Phi,z_0}^{\mathcal{M}}(D)$ is compact in D.