Zolkina I. S.
Kolomiiets T. Yu.
(Zhytomyr, Ukraine)

THE NORMAL DISTRIBUTION LAW OF RANDOM QUATERNIONS AND ITS APPLICATION

In recent years, research on quaternion normal distribution has been devoted to works in which authors study various approaches to developing and describing this distribution using complex mathematical methods [1-3]. This paper presents a visual model of the normal distribution of random quaternions (under specified conditions), which is a generalization of the three-sigma rule of the classical normal distribution to the case of standard deviations of quaternions.

Let us have a finite sample from a set of random quaternions Q:

$$Q_m = a + bi + cj + dk, \quad m = 1, 2, \ldots, n,$$

whose components $a, b, c, d \in \mathbb{R}$ are normally distributed random variables with corresponding probabilities $p_1, p_2, p_3, p_4 \in \mathbb{R}$. Then we write the probabilities of quaternions appearing in the sample as quaternion probabilities:

$$P_m = p_1 + p_2 i + p_3 j + p_4 k, \quad p_1, p_2, p_3, p_4 \in \mathbb{R}, \quad m = 1, 2, \ldots, n.$$

The *mathematical expectation* of random quaternions Q has two formulas:

$$M_R(Q) = \sum_{m=1}^{n} Q_m P_m; \quad M_L(Q) = \sum_{m=1}^{n} P_m Q_m. \tag{1}$$

Using the right-hand and left-hand mathematical expectations (1), we obtain four formulas for the variances of random quaternions Q:

$$D_{RR}(Q) = \sum_{m=1}^{n} \left(Q_m - M_R(Q)\right)^2 P_m; D_{RL}(Q) = \sum_{m=1}^{n} \left(Q_m - M_L(Q)\right)^2 P_m;$$
$$D_{LL}(Q) = \sum_{m=1}^{n} P_m \left(Q_m - M_L(Q)\right)^2; D_{LR}(Q) = \sum_{m=1}^{n} P_m \left(Q_m - M_R(Q)\right)^2. \tag{2}$$

According to (2), we have four standard deviations:

$$\sigma_{RR}(Q) = \sqrt{D_{RR}(Q)}; \quad \sigma_{RL}(Q) = \sqrt{D_{RL}(Q)};$$
$$\sigma_{LL}(Q) = \sqrt{D_{LL}(Q)}; \quad \sigma_{LR}(Q) = \sqrt{D_{LR}(Q)}. \tag{3}$$

Let us consider a sample of unique random quaternions in the sense that each quaternion will occur only once in this sample, i.e., the probabilities of encountering all quaternions are equal. In this case, we will write the quaternion-valued probabilities as:

$$p_m = p(1 + i + j + k), \quad m = 1, 2, \ldots, n, \quad p \in \left[0; \frac{1}{2}\right], \quad p = \frac{1}{2n}. \qquad (4)$$

Writing down formulas (1) – (3) and taking into account (4), we construct a visualization of the normal distribution of random quaternions (Fig. 1). The model is implemented in the Python programming language.

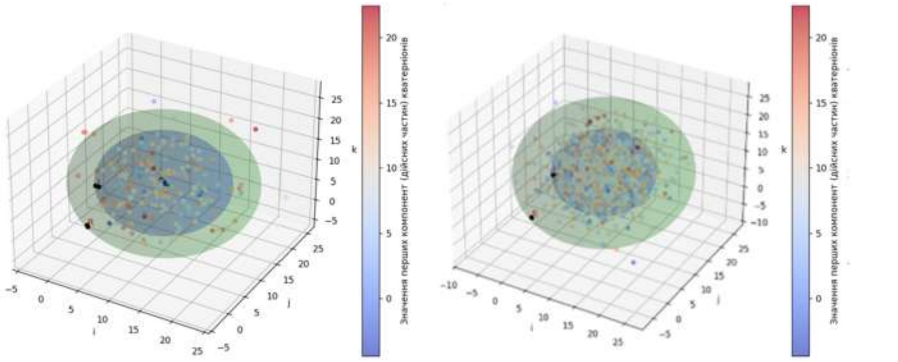

Рис. 1: Normal distribution of 200 and 1000 quaternions at $M = 10, \sigma = 5$.

Empirically, we constructed distributions of random quaternions with volumes of 100, 150, 200, 500, and 1000 points at $M = 10, \sigma = 5$. Based on the constructed models, we concluded that the visual model of the normal distribution of random quaternions looks like two spheres, with the left sphere usually closer to the center of dispersion and the right sphere further from the center of dispersion. Fig.1 shows that the larger the sample size, the greater the number of quaternions located closer to the center of dispersion. In this regard, we formulate the rule of standard deviations of quaternions as follows: almost 68% of quaternion points fall within the sphere with the smaller radius (i.e., the left standard deviations play the role of 1σ), and almost 99,7% of quaternions fall within the sphere with the larger radius (i.e., the right standard deviations play the role of 3σ). Elements located outside the larger sphere are at distances greater than the radii of standard deviations from the center of dispersion and therefore fall outside the 3σ limits.

We can see that the model (Fig.1) has similarities with real physical objects – spherical star clusters studied in astrophysics. This makes it possible to apply the obtained model, for example, to simulate the interaction of stars in clusters or as initial conditions for the evolution of star motion over time.

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