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RIEMANN ZETA FUNCTION OF A VARIABLE FROM COMMUTATIVE ALGEBRA

Abstract. We introduce and study the extension of the Riemann zeta function to commutative algebras with identity.

Suppose $\mathbb{A}$ is a d-dimensional commutative algebra over a field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) with a basis $e_1, e_2, \ldots, e_d$ and e_1 is the identity of $\mathbb{A}$.

Suppose $\mathbb{A}$ has non-trivial idempotents $i_1, i_2, \ldots, i_d$ such that $i_k i_m = 0$, $k \neq m$, and $\sum_{k=1}^{d} i_k = 1$.

Denote by $I_m = \{a i_m | a \in \mathbb{A}\}$ the principal ideal generated by i_m, $m = 1, \ldots, d$. It is easily seen that $\mathbb{A}$ can be decomposed in the direct sum (the Pierce decomposition)

$\mathbb{A} = I_1 \oplus I_2 \oplus \ldots \oplus I_d$.

Lemma 1. *Idempotents $i_1, i_2, \ldots, i_d$ are linearly independent.*

Lemma 2. *If $b \in I_m$ then there exists $k \in \mathbb{K}$ such that $b = k i_m$, i.e., the ideal I_m can be represented as follows $I_m = \{k i_m | k \in \mathbb{K}\}$.*

Theorem 1. *For any $a \in \mathbb{A}$ we have the following decomposition*

$$a = \sum_{m=1}^{d} k_m i_m, \quad k_m \in \mathbb{K}.$$

It is easily verified that $a^l = \sum_{m=1}^{d} k_m^l i_m$.

Let us define $a \in \mathbb{A}$ power of a natural number.

For $a = \sum_{m=1}^{d} k_m i_m$ and $n \in \mathbb{N}$, we have

$$n^a = \exp(a \ln n) = \sum_{l=0}^{\infty} \frac{a^l (\ln n)^l}{l!} = \sum_{m=1}^{d} \sum_{l=0}^{\infty} \frac{k_m^l (\ln n)^l}{l!} i_m = \sum_{m=1}^{d} n^{k_m} i_m.$$

The zeta function of $a\in\mathbb{A}$ variable

For $a=\sum_{m=1}^{d}k_m i_m$

$$\zeta(a)=\sum_{n=0}^{\infty}n^{-a}=\sum_{n=0}^{\infty}\sum_{m=1}^{d}n^{-k_m}i_m=\sum_{m=1}^{d}\sum_{n=0}^{\infty}n^{-k_m}i_m=\sum_{m=1}^{d}\zeta\left(k_m\right)i_m.$$

Conclusion.

If $\mathbb{K}=\mathbb{R}$ the zeta function has only the trivial zeros at points $k_m=-2l_m$, $l_m\in\mathbb{N}$, $m=1,2,\ldots,d$.

If $\mathbb{K}=\mathbb{C}$ the $\mathbb{A}$ algebra Riemann hyposesis is as follows:

Non-trivial zeros of the zeta function of variable $a\in\mathbb{A}$ are located at $a=\frac{1}{2}+i\Sigma$, where $\Sigma=\sum_{m=1}^{d}\sigma_m i_m$, $m=1,2,\ldots,d$ and $\frac{1}{2}+i\sigma_m$ is zero of the complex zeta function $\zeta(z)$.

R E F E R E N C E S

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