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AN EXTREMAL PROBLEM FOR A MOSAIC SYSTEM OF POINTS

In the geometric theory of functions of a complex variable, the well-known direction is related to the estimates of the products of the inner radii of pairwise nonoverlapping domains. This direction is called extreme problems in classes of pairwise nonoverlapping domains [1]. One of the problems of this type is considered in the present work.

Let the numbers $n, m, d \in \mathbb{N}$ be fixed.

The system of points $A_{n,m} = \{a_{k,p} \in \mathbb{C} : k = \overline{1,n}, p = \overline{1,m}\}$, is called an (n,m)-ray system of points, if, for all $k = \overline{1,n}$, the following relations hold:

$$0 < |a_{k,1}| < \ldots < |a_{k,m}| < \infty;$$

$$\arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,m} =: \theta_k;$$
$$0 = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} := 2\pi.$$

For such systems of points, let us consider the following quantities:

$$\alpha_k = \frac{1}{\pi}\left[\theta_{k+1} - \theta_k\right],\ k = \overline{1,n},\ \alpha_{n+1} := \alpha_1,\ \alpha_0 := \alpha_n, \sum_{k=1}^{n} \alpha_k = 2.$$

For any (n,m)-equiangular ray system of points $A_{n,m} = \{a_{k,p}\}$, we consider the following controlling functional

$$M\left(A_{n,m}\right) = \prod_{k=1}^{n}\prod_{p=1}^{m}\left[\chi\left(|a_{k,p}|^{\frac{1}{\alpha_k}}\right)\cdot\chi\left(|a_{k,p}|^{\frac{1}{\alpha_{k-1}}}\right)\right]^{\frac{1}{2}}\cdot|a_{k,p}|,$$

where $\chi(t) = \frac{1}{2}\cdot\left(t + t^{-1}\right)$.

Consider the system of angular domains:

$$P_k\left(A_{n,m}\right) = \{w \in \mathbb{C} : \theta_k < \arg w < \theta_{k+1}\}, \quad k = \overline{1,n}.$$

For a fixed number $\beta, R \in \mathbb{R}^+$, $0 < \beta < \frac{2\pi}{n}$, consider the unique branch of the multibranch analytic function

$$z_k(w) = \frac{i}{R^{\frac{1}{\alpha_k}}\cdot\sin\frac{\beta}{\alpha_k}}\cdot\left(-\left(e^{-i\theta_k}w\right)^{\frac{1}{\alpha_k}} + R^{\frac{1}{\alpha_k}}\cdot\cos\frac{\beta}{\alpha_k}\right). \tag{1}$$

For each $k = \overline{1,n}$, it realizes the one-sheet conformal mapping of the domain P_k onto the right half-plane $\mathrm{Re} z > 0$.

For each $k = \overline{1,n}$ we denote

$$\begin{aligned}
&\Omega_j^{(k)} := \left\{z : |z - i\varrho_j| = r_j, 0 \le \arg z \le \tfrac{\pi}{2}, \varrho_j \in \mathbb{R}, r_j \in \mathbb{R}^+\right\}, j = \overline{1,m},\\
&\Omega_j^{(k)} := \left\{z : |z - i\varrho_j| = r_j, -\tfrac{\pi}{2} \le \arg z \le 0, \varrho_j \in \mathbb{R}, r_j \in \mathbb{R}^+\right\},\\
&j = \overline{m+1, 2m},
\end{aligned} \tag{2}$$

where

$$\varrho_1 + r_1 > \varrho_2 + r_2 > \ldots > \varrho_{2m} + r_{2m}.$$

Let $\{b_k\}_{k=1}^{n} \subset \mathbb{C}$ be a set of points such that

$$b_k \in P_k,\ \arg b_k - \theta_k = \beta,\ |b_k| = R,\ k = \overline{1,n}.$$

Let, for each fixed k, $k=\overline{1,n}$, $\left\{L_j^{(k)}\right\}_{j=1}^{2m}$ – be a collection of curves such that

$$\begin{aligned} &L_j^{(k)} \subset \overline{P_k},\ b_k \in L_j^{(k)},\ j=\overline{1,2m}, \\ &a_{k,p} \in L_{m-p+1}^{(k-1)},\ a_{k,p} \in L_{m+p}^{(k)},\ p=\overline{1,m}, \\ &z_k:\ L_j^{(k)} \to \Omega_j^{(k)},\ j=\overline{1,2m}. \end{aligned} \qquad (3)$$

It is easy to see from relations (1), (2), (3) that

$$z_k\left(b_k\right)=1,\ z_k\left(a_{k+1,p}\right)=i\lambda_p,\ z_k\left(a_{k,p}\right)=-i\lambda_{m+p},$$

$$a_{n+1,p}:=a_{1,p},\ \lambda_t>0, t=\overline{1,2m}, k=\overline{1,n}, p=\overline{1,m}.$$

For each $k=\overline{1,n}$ we denote the corresponding systems of points by

$$D_{2m,d}^{(k)}=\left\{c_{j,s}^{(k)} \in L_j^{(k)}:\ 0<\left|\arg z_k\left(c_{j,1}^{(k)}\right)\right|<\left|\arg z_k\left(c_{j,2}^{(k)}\right)\right|<\right.$$

$$\left.<\ldots<\left|\arg z_k\left(c_{j,d}^{(k)}\right)\right|<\frac{\pi}{2},\ j=\overline{1,2m},\ s=\overline{1,d}\right\}.$$

The system of points

$$AD_{n,m,d}=\bigcup_{k=1}^{n} D_{2m,d}^{(k)} \bigcup A_{n,m}$$

will be called mosaic.

For any mosaic system of points $AD_{n,m,d}$, we consider the following "controlling" functional

$$\mu\left(AD_{n,m,d}\right):=\prod_{k=1}^{n}\left(\left|a_{k,p}\right| \cdot\left|c_{j,s}^{(k)}\right|^{2d}\right)^{1-\frac{1}{\alpha_k}}.$$

For the mosaic point system, the valid result obtained is [2].

REFERENCES

1. A.K. Bakhtin, G.P. Bakhtina, Yu.B. Zelinskii *Topological-algebraic structures and geometric methods in complex analysis*, Inst. Math. NAS Ukraine, Kiev, (2008).

2. Targonskii A., Bondar S. *An extremal problem for a mosaic system of points in the case of an additional set of points on a circle*, Journal of Mathematical Sciences. **284**(1), 400–409 (2024).