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On boundary distortion estimates of homeomorphisms with a fixed point

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Abstract. The article is devoted to the study of homeomorphisms that distort the modulus of families of paths according to the Poletsky inequality type. We consider the case when the majorant, corresponding to the distortion of the modulus, has finite mean oscillation at any point or satisfies Lehto integral divergence condition. We have obtained the boundary Hölder continuity of such mappings, provided that the image of at least one inner point is located at a fixed distance from the boundary of the mapped domain. In particular, this condition holds for fixed-point mappings. The manuscript deals with cases of good boundaries and domains with prime ends.

Анотація. Стаття присвячена вивченню гомеоморфізмів, які спотворюють модуль сімей кривих по типу нерівності Полецького.

Розглядається випадок, коли мажоранта, що відповідає спотворенню модуля, має скінченне середнє коливання в будь-якій точці, або задовольняє умову інтегральної розбіжності типу Лехто. Нами отримана межова неперервність за Гельдером таких відображень за умови, що образ принаймні однієї внутрішньої точки знаходиться на фіксованій відстані від межі відображеної області. Зокрема, вказана умова виконується для відображень з фіксованою точкою. Ми окремо розглядаємо гарні межі та області з простими кінцями.

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1. INTRODUCTION

This article is devoted to the boundary behavior of mappings with a finite distortion. In particular, we deal with mappings associated with the distortion of the modulus of families of paths (see, e.g., [14]). It should be noted that the boundary behavior occupies a special place in theory of quasiregular mappings and mappings with finite distortion. In turn, the qualitative behavior of mappings on the boundary is associated with the Lipschitz and Hölder inequalities, for example, [4, 13, 18]. Several recent publications by the first co-author are also devoted to above problems. In particular, we have obtained boundary estimates of the distortion under mappings satisfying weighted Poletsky inequality (see, e.g., [1, 21, 2], cf. [20]). We have considered here domains with different geometry, including domains locally connected on their boundary, domains with the Poincare inequality, domains with prime ends, etc. Besides that, we were interested in the case when the majorant Q responsible for the distortion of the modulus of families of paths has finite means over infinitesimal balls. In this paper we continue our research on this topic, considering more general conditions on Q. Namely, we investigate the condition of a finite mean oscillation, or divergence of the Lehto type integral. Also one of the important necessary conditions is the requirement that the image of some point under the mapping does not approach the boundary of the image domain, cf. [2].

In what follows, D is a domain in $\mathbb{R}^n$, $n\ge 2$,

$$B(x_0,r)=\{x\in\mathbb{R}^n : |x-x_0|<r\}, \qquad \mathbb{B}^n=B(0,1),$$
$$S(x_0,r)=\{x\in\mathbb{R}^n : |x-x_0|=r\}, \qquad \mathbb{S}^{n-1}=S(0,1),$$
$$\Omega_n=m(\mathbb{B}^n), \qquad \omega_{n-1}=\mathcal{H}^{n-1}(\mathbb{S}^{n-1}),$$

m is a Lebesgue measure in $\mathbb{R}^n$, $\mathcal{H}^{n-1}$ is a $(n-1)$-measured Hausdorff measure,

$$A(x_0,r_1,r_2):=\{x\in\mathbb{R}^n : r_1<|x-x_0|<r_2\},$$

and $M(\Gamma)$ denotes the *conformal modulus of the family of paths* Γ (see [22]). Let $Q\colon\mathbb{R}^n\to[0,\infty]$ be a Lebesgue-measurable function equal to zero outside D. Consider the following concept, see [14, section 7.6]. We say that a mapping $f\colon D\to\overline{\mathbb{R}^n}$ is a *ring Q-mapping at a point* $x_0\in\overline{D}$, $x_0\ne\infty$, if the condition

$$M\big(f(\Gamma(S(x_0,r_1),S(x_0,r_2),D))\big)\le\int\limits_{A(x_0,r_1,r_2)} Q(x)\cdot\eta^n(|x-x_0|)\,dm(x) \qquad (1.1)$$

is fulfilled for some $r_0=r(x_0)>0$ and arbitrary $0<r_1<r_2<r_0$, where $\eta:(r_1,r_2)\to[0,\infty]$ is an arbitrary nonnegative Lebesgue measurable

function satisfying the inequality

$$\int\limits_{r_1}^{r_2} \eta(r)dr \geq \ 1. \tag{1.2}$$

It can be shown that all quasiconformal and quasiregular mappings satisfy relation (1.1), since they also satisfy the relation

$$M\big(f(\Gamma(S(x_0,r_1),S(x_0,r_2),D))\big) \leq K\cdot M\big(\Gamma(S(x_0,r_1),S(x_0,r_2),D)\big)$$

while $Q=K=\text{const}$ (see, for example, [17, 22]).

Following [3], a domain G in $\mathbb{R}^n$ is called a *quasiextremal distance domain* (short. *QED-domain*), if there is a number $A_0\geq 1$, such that the inequality

$$M\big(\Gamma(E,F,\mathbb{R}^n)\big) \leq A_0\cdot M\big(\Gamma(E,F,G)\big) \tag{1.3}$$

holds for any continua $E,F\subset G$. Given sets $A,B\subset\mathbb{R}^n$, we put

$$\operatorname{diam} A=\sup\limits_{x,y\in A}|x-y|, \qquad \operatorname{dist}(A,B)=\inf\limits_{x\in A,y\in B}|x-y|.$$

Sometimes instead of $\operatorname{diam} A$ and $\operatorname{dist}(A,B)$ we also write $d(A)$ and $d(A,B)$, respectively.

Recall that a domain $D\subset\mathbb{R}^n$ is called *locally connected at the point* $x_0\in\partial D$, if for any neighborhood U of the point x_0 there is a neighborhood $V\subset U$ of this point such that $V\cap D$ is connected. A domain D is *locally connected on* ∂D, if D is locally connected at every point $x_0\in\partial D$.

Below the *spherical (chordal)* metrics in $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ is defined as follows:

$$h(E):=\sup\limits_{x,y\in E}h(x,y), \qquad h(A,B)=\inf\limits_{x\in A,y\in B}h(x,y), \tag{1.4}$$

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}},$$

$$h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2}\sqrt{1+{|y|}^2}}, \quad x\neq\infty\neq y.$$

Given a function $Q\colon D\to[0,\infty]$, $Q(x)\equiv 0$ in $\mathbb{R}^n\setminus D$, denote by $q_{x_0}(r)$ the integral average of the function $Q(x)$ under the sphere $|x-x_0|=r$, i.e.,

$$q_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r}Q(x)dS,$$

where dS is the area element of S, and ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$.

Given a domain $D\subset {\mathbb R}^n$, $n\geq 2$, points $x_0\in \partial D$, $x_0\ne\infty$, $a\in D$, numbers $A_0>0$ and $\delta>0$, and a function $Q\colon D\rightarrow[0, \infty]$ denote by

$$\mathfrak{S}^{A_0}_{Q, a, \delta}(D, x_0)$$

the family of all homeomorphisms $f\colon D\rightarrow {\mathbb R}^n$ satisfying conditions (1.1)–(1.2) at x_0 such that $D_f^{\,\prime}=f(D)$ satisfies the condition (1.3) with $G\mapsto D_f^{\,\prime}$, and $h(f(a), \partial D_f^{\,\prime})\geq \delta.$ The following statement is true.

Theorem 1.1. *Assume that, the following conditions hold:*
1) *D is locally is connected at its boundary,*
2) *there is $r_0^{\prime}=r_0^{\prime}(x_0)>0$ such that the set $B(x_0, r)\cap D$ is connected for all $0<r<r_0^{\prime}$;*
3) *there is $\delta_0=\delta_0(x_0)>0$ such that*

$$\int\limits_0^{\delta_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}=\infty. \tag{1.5}$$

Then any $f\in \mathfrak{S}^{A_0}_{Q, a, \delta}(D, x_0)$ has a continuous extension to x_0, in addition, there is $\varepsilon_0=\varepsilon_0(x_0)>0$ and a number $\alpha>0$ such that the inequality

$$h(f(x), f(y))\leq\alpha\cdot\exp\left\{-\left(A_0\right)^{-\frac{1}{n-1}}\int\limits_{|x-x_0|}^{\varepsilon_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}\right\}$$

holds for all $x, y\in B(x_0, \varepsilon_0(x_0))\cap D$ such that $|x-x_0|\geq |y-x_0|$ and all $f\in \mathfrak{S}^{A_0}_{Q, a, \delta}(D, x_0)$.

Remark 1.2. In particular, condition 1) of Theorem 1.1 is fulfilled if D is a convex domain.

Following [8], we say that a function $\varphi\colon D\rightarrow{\mathbb R}$ has *finite mean oscillation* (*FMO*) at a point $x_0\in D$ if

$$\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0, \varepsilon)}|\varphi(x)-\widetilde{\varphi_{\varepsilon}}|\,dm(x)<\infty,$$

where

$$\widetilde{\varphi_{\varepsilon}}=\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0, \varepsilon)}\varphi(x)\,dm(x)$$

is the mean of the function $\varphi(x)$ over the ball $B(x_0, \varepsilon)$. Note that FMO is not BMO_{loc} ([14, p. 211]). It is well known [9] that

$$L^{\infty}(D)\subset BMO(D)\subset L^p_{\rm loc}(D), \qquad 1\leq p<\infty,$$

but $FMO(D)\not\subset L^p_{\rm loc}(D)$ for any $p>1$ (see [14]).

Theorem 1.3. *Assume that, under conditions of Theorem 1.1, instead of* (1.5) *the following condition holds:*

$$Q \in FMO(x_0).$$

Then there is $\varepsilon_0 = \varepsilon_0(x_0) > 0$ *and a number* $C > 0$ *such that the inequality*

$$h(f(x), f(y)) \le C \left\{ \frac{1}{\log \frac{1}{|x-x_0|}} \right\}^p \tag{1.6}$$

holds for some a number $p := \left(\frac{\omega_{n-1}}{A_0 2^n C_1}\right)^{\frac{1}{n-1}}$*, where* $C_1 > 0$ *depends only on the function* Q*, and all* $x, y \in B(x_0, \varepsilon(x_0)) \cap D$ *such that* $|x - x_0| \ge |y - x_0|$ *and all* $f \in \mathfrak{S}^{A_0}_{Q,a,\delta}(D, x_0)$*.*

Theorems 1.1 and 1.3 have the corresponding counterparts also for domains with bad boundaries, see below. The definition of a prime end used below may be found in [7, 6], cf. [11, 16].

Let ω be an open set in $\mathbb{R}^k$, $k = 1, \ldots, n-1$. A continuous mapping $\sigma\colon \omega \to \mathbb{R}^n$ is called a *k-dimensional surface* in $\mathbb{R}^n$. A *surface* is an arbitrary $(n-1)$-dimensional surface σ in $\mathbb{R}^n$. A surface σ is called *a Jordan surface*, if $\sigma(x) \ne \sigma(y)$ for $x \ne y$.

In what follows, we will use σ instead of $\sigma(\omega) \subset \mathbb{R}^n$, $\overline{\sigma}$ instead of $\overline{\sigma(\omega)}$ and $\partial\sigma$ instead of $\overline{\sigma(\omega)} \setminus \sigma(\omega)$. A Jordan surface $\sigma\colon \omega \to D$ is called a *cut* of D, if σ separates D, that is $D \setminus \sigma$ has more than one component, $\partial\sigma \cap D = \varnothing$ and $\partial\sigma \cap \partial D \ne \varnothing$.

A sequence of cuts $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ in D is called *a chain*, if:

(i) the set σ_{m+1} is contained in exactly one component d_m of the set $D \setminus \sigma_m$, wherein $\sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m)$;

(ii) $\bigcap\limits_{m=1}^{\infty} d_m = \varnothing$.

Two chains of cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called *equivalent*, if for each $m = 1, 2, \ldots$ the domain d_m contains all the domains d'_k, except for a finite number, and for each $k = 1, 2, \ldots$ the domain d'_k also contains all domains d_m, except for a finite number.

The *end* of the domain D is the class of equivalent chains of cuts in D. Let K be the end of D in $\mathbb{R}^n$, then the set

$$I(K) = \bigcap_{m=1}^{\infty} \overline{d_m}$$

is called *the impression of the end K*. One may to prove that, $I(P) \subset \partial D$ (see, e.g., [11, Proposition 1]).

Following [16], we say that the end K is *a prime end*, if K contains a chain of cuts $\{\sigma_m\}$ such that

$$\lim_{m\to\infty} M\big(\Gamma(C,\sigma_m,D)\big)=0 \tag{1.7}$$

for some continuum C in D. We will use below the following notation: the set of prime ends corresponding to the domain D is denoted by E_D, and the completion of the domain D by its prime ends is denoted by $\overline{D}_P$.

Consider the following definition, which goes back to Näkki [16], see also [11]. We say that the boundary of the domain D in ${\Bbb R}^n$ is *locally quasiconformal*, if each point $x_0\in\partial D$ has a neighborhood U in ${\Bbb R}^n$, which can be mapped by a quasiconformal mapping φ onto the unit ball ${\Bbb B}^n\subset{\Bbb R}^n$ so that $\varphi(\partial D\cap U)$ is the intersection of ${\Bbb B}^n$ with the coordinate hyperplane.

The sequence of cuts σ_m, $m=1,2,\ldots$, is called *regular*, if $\overline{\sigma_m}\cap\overline{\sigma_{m+1}}=\varnothing$ for $m\in{\Bbb N}$ and, in addition, $d(\sigma_m)\to 0$ as $m\to\infty$. If the end K contains at least one regular chain, then K will be called *regular*.

We say that a bounded domain D in ${\Bbb R}^n$ is *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in ${\Bbb R}^n$, and, besides that, every prime end in D is regular.

Note that space $\overline{D}_P=D\cup E_D$ admits a metric defined in the following way. If $g\colon D_0\rightarrow D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y\in\overline{D}_P$ we put:

$$\rho(x, y):=|g^{-1}(x)-g^{-1}(y)|, \tag{1.8}$$

where the elements $g^{-1}(x)$ and $g^{-1}(y)$, $x, y\in E_D$, are to be understood as some (single) boundary points of the domain D_0. It is easy to verify that ρ in (1.8) is a metric on $\overline{D}_P$, and that the topology on $\overline{D}_P$, defined by such a method, does not depend on the choice of the map g with the indicated property.

We say that a sequence $x_m\in D$, $m=1,2,\ldots$, converges to a prime end $P\in E_D$ as $m\rightarrow\infty$, if for any $k\in {\Bbb N}$ all elements x_m belong to d_k except for a finite number. Here d_k denotes a sequence of nested domains corresponding to the definition of the prime end P.

Given a domain $D\subset {\Bbb R}^n$, $n\geq 2$, points $P_0\in E_D$, $a\in D$, numbers $A_0>0$ and $\delta>0$, and a function $Q\colon D\rightarrow [0, \infty]$ denote by

$$]\mathfrak{S}^{A_0}_{Q, a, \delta}(D, P_0)$$

the family of all homeomorphisms $f\colon D\rightarrow {\Bbb R}^n$ satisfying (1.1)–(1.2) at any $x_0\in I(P_0)$ (where $I(P_0)$ is the impression of P_0) such that $D^{\,\prime}_f=f(D)$ satisfies (1.3), $G\mapsto D^{\,\prime}_f$, and $h(f(a), \partial D^{\,\prime}_f)\geq \delta$. The following statement holds.

Theorem 1.4. *Assume that, D is a regular domain and the following conditions are fulfilled:*

1) *for any $y_0\in\partial D$ there is $r_0^{\prime}=r_0^{\prime}(y_0)>0$ such that $B(y_0, r)\cap D$ is finitely connected for all $0<r<r_0^{\prime}$, and, for each connected component K of the set $\overline{B(y_0, r)}\cap D$ the following condition is fulfilled: any pair of points $x, y\in K$ can be joined by a path $\gamma\colon [a, b]\rightarrow {\mathbb R}^n$ such that*
$$|\gamma|\subset K\cap \overline{B(x_0, \max\{|x-y_0|, |y-y_0|\})}$$
and
$$|\gamma|=\{x\in {\mathbb R}^n: \exists\,t\in [a, b]: \gamma(t)=x\};$$
2) *there is $\delta_0=\delta_0(x_0)>0$ such that*
$$\int\limits_0^{\delta_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}=\infty. \tag{1.9}$$

Then, for any $P\in E_D$ there is $y_0\in \partial D$ such that $I(P)=\{y_0\}$, where $I(P)$ denotes the impression of P. Furthermore, f has a limit as $x\rightarrow P_0$, in addition, there exists there exist a neighborhood U of P_0 and a number $\alpha>0$ such that the relation
$$h(f(x),f(y))\leq\alpha_n\cdot\exp\left\{-\left(A_0\right)^{-\frac{1}{n-1}}\int\limits_{|x-x_0|}^{\varepsilon_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}\right\}$$
holds for any $x, y\in U\cap D$ such that $|x-x_0|\geq|y-x_0|$ and all f from $\mathfrak{S}_{Q,a,\delta}^{A_0}(D, P_0)$, where $I(P_0)=\{x_0\}$.

Theorem 1.5. *Assume that, under conditions of Theorem 1.4, instead of (1.9) the following condition holds: $Q\in FMO(x_0)$ for any $x_0\in I(P_0)$. Then f has a limit as $x\rightarrow P_0$, in addition, there exists a neighborhood U of P_0 and a number $C>0$ such that the inequality*
$$h(f(x),f(y))\leq C\left\{\frac{1}{\log\frac{1}{|x-x_0|}}\right\}^p$$
holds for some a number $p:=\left(\frac{\omega_{n-1}}{A_02^nC_1}\right)^{\frac{1}{n-1}}$, where $C_1>0$ depends only on the function Q, all $x, y\in U\cap D$ such that $|x-x_0|\geq|y-x_0|$, and all $f\in\mathfrak{S}_{Q,a,\delta}^{A_0}(D, P_0)$, where $I(P_0)=\{x_0\}$.

Remark 1.6. In particular, condition 1) of Theorem 1.4 is fulfilled if, for any $y_0\in\partial D$ there is $r_0^{\prime}=r_0^{\prime}(y_0)>0$ such that the set $B(y_0, r)\cap D$ is finitely connected for all $0<r<r_0^{\prime}$, and any component K of the set $B(y_0, r)\cap D$ is convex.

Indeed, let $x, y \in K$. Let us join x and y with a segment γ inside K (this is possible since the connected open set K is also path connected, see, e.g., [14, Corollary 13.1]). Assume for definiteness that $|x-y_0| \geq |y-y_0|$. Since the ball $\overline{B(x, |x-y_0|)}$ is convex, the entire segment γ belongs to $B(x, |x-y_0|)$. Then γ is the desired path.

2. MAPPINGS OF DOMAINS LOCALLY CONNECTED ON THE BOUNDARY

A domain R in $\overline{\mathbb{R}^n}$, $n \geq 2$, is called *a ring*, if $\overline{\mathbb{R}^n} \setminus R$ consists of exactly two components E and F. In this case, we write: $R = R(E, F)$. The following statement is true, see [14, ratio (7.29)].

Proposition 2.1. *If $R = R(E, F)$ is a ring, then*

$$M\big(\Gamma(E, F, \overline{\mathbb{R}^n})\big) \geq \frac{\omega_{n-1}}{\left(\log \frac{2\lambda_n^2}{h(E)h(F)}\right)^{n-1}},$$

where $\lambda_n \in [4, 2e^{n-1})$, $\lambda_2 = 4$ and $\lambda_n^{1/n} \to e$ as $n \to \infty$, and $h(E)$ denotes the chordal diameter of the set E defined in (1.4).

The following statement holds.

Lemma 2.2. *Assume that, under conditions of Theorem 1.1, instead of the condition* 3) *the following holds: there are $p < n$ and a Lebesgue measurable function $\psi\colon (\varepsilon, \varepsilon_0) \to [0, \infty]$, $\varepsilon \in (0, \varepsilon_0)$, such that the relation*

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x) \cdot \psi^n(|x-x_0|)\, dm(x) \leq K_0 \cdot I^p(\varepsilon, \varepsilon_0) \tag{2.1}$$

holds as $\varepsilon \to 0$ for some $K_0 > 0$, where

$$0 < I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \tag{2.2}$$

for sufficiently small $\varepsilon > 0$. Assume that $|x-x_0| \geq |y-y_0|$ and that $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$. Then there is a number $\lambda_n > 0$ such that the inequality

$$h(f(x), f(y)) \leq \frac{\lambda_n^2}{\delta} \exp\left\{-\left(\frac{\omega_{n-1}}{A_0 K_0}\right)^{\frac{1}{n-1}} I^{\frac{n-p}{n-1}}(|x-x_0|, \varepsilon_0)\right\} \tag{2.3}$$

holds for all $x, y \in B(x_0, \varepsilon_0(x_0)) \cap D$ and all $f \in \mathfrak{S}_{Q,a,\delta}^{A_0}(D, x_0)$.

Proof. Let us now make the following auxiliary considerations. If ∂D contains at least one finite point other than x_0, denote it by y_0. Otherwise, put $y_0 := \infty$. Since D is locally connected on its boundary, the point a

may be joined to the point y_0, which belongs to D entirely, excluding the point y_0 itself (see [14, Proposition 13.2]). Let us denote this path by E. Without limiting the generality, we may assume that $E \subset D \setminus B(x_0, \varepsilon_0)$, where ε_0 is a number defined above, and also that E is a Jordan path.

Now fix $f \in \mathfrak{S}^{A_0}_{Q,a,\delta}(D, x_0)$ and $x, y \in B(x_0, \varepsilon_0)$. Note that

$$C(y_0, f) \subset \partial D'_f, \qquad\qquad D'_f := f(D),$$

where $C(y_0, f)$ is the cluster set of f at the point y_0 (see, e.g., [14, Proposition 13.5]). Let us prove the existence of a point $a_f \in E$, such that

$$h(f(a), f(a_f)) \geq (1/2) \cdot h(f(a), \partial D'_f) \geq \delta/2. \tag{2.4}$$

Indeed, let $z_k \in E$, $k = 1, 2 \ldots$, be an arbitrary sequence such that $z_k \to y_0$ as $k \to \infty$. Since the space $\overline{\mathbb{R}^n}$ is compact, we may assume that the sequence $f(z_k)$ also converges to some y'_0 as $k \to \infty$ with a respect to the chordal metric. Since $C(y_0, f) \subset \partial D'_f$, we obtain that $y'_0 \in \partial D'_f$. Also, since $f(z_k) \to y'_0$ as $k \to \infty$, for the number $\delta/2$, we may find $k_0 = k_0(f) \in \mathbb{N}$ such that

$$h(f(z_k), y'_0) < \delta/2 \quad \forall k \geq k_0 \tag{2.5}$$

Put $a_f := z_{k_0}$. Then, by the triangle inequality, relation (2.5), and the definition of the class $\mathfrak{S}^{A_0}_{Q,a,\delta}(D, x_0)$, we obtain that

$$\begin{aligned} \delta &\leq h(f(a), \partial D'_f) \\ &\leq h(f(a), y'_0) \\ &\leq h(f(a), f(a_f)) + h(f(a_f), y'_0) \\ &\leq h(f(a), f(a_f)) + \delta/2, \end{aligned} \tag{2.6}$$

or, transferring $\delta/2$ to the left side (2.6),

$$\delta/2 \leq h(f(a), \partial D'_f) \leq h(f(a), f(a_f)).$$

The latter relation proves (2.4).

Now, let A_f be a continuum in D, which is part of E from the point a to the point a_f. Also, let $x, y \in B(x_0, \varepsilon_0)$. By the definition of ε_0,

$$A_f \subset D \setminus B(x_0, \varepsilon_0). \tag{2.7}$$

Since the points of the sphere

$$S(x_0, r) \cap D, \qquad 0 < r < r'_0,$$

are accessible from the domain D by using some path γ, and the set $B(x_0, r) \cap D$ is connected for all $0 < r < r'_0$, the points x and y may be joined by a path K, which belongs entirely to the ball $\overline{B(x_0, |x - x_0|)}$ and lies in D. This a path K may also be considered Jordanian.

Note that, the paths $f(K)$ and $f(A_f)$ are also Jordanian and do not split $\mathbb{R}^n$. Indeed, for $n\geq 3$ the set

$$f(|K|)\cup f(A_f)$$

has a topological dimension 1 as a union of two closed sets of topological dimension 1 (see [5, Theorem III.2.3]). Then $f(|K|)\cup f(A_f)$ do not split $\mathbb{R}^n$ (see [5, Corollary 1.5.IV]).

Let now $n=2$. According to Antoine's theorem on the absence of wild arcs (see [10, Theorem II.4.3]), there exists a homeomorphism

$$\varphi\colon \mathbb{R}^2\rightarrow \mathbb{R}^2,$$

which maps $f(|K|)$ on some segment I. It follows that, any points $x, y\in \mathbb{R}^2\setminus f(|K|)$ may be joined by the path γ in $\mathbb{R}^2\setminus f(|K|)$. Reasoning similarly, it may be shown that, any points $x, y\in \mathbb{R}^2\setminus (f(|K|)\bigcup f(A_f))$ may be joined by the path γ in $\mathbb{R}^2\setminus (f(|K|)\bigcup f(A_f))$. Therefore,

$$R=R(f(|K|), f(A_f))$$

is a ring domain. In this case, put

$$\Gamma=\Gamma(f(|K|), f(A_f), \mathbb{R}^n).$$

Then by Proposition 2.1

$$M\big(\Gamma(f(|K|), f(A_f), \overline{\mathbb{R}^n})\big)\geq \frac{\omega_{n-1}}{\left(\log\frac{2\lambda_n^2}{h(f(|K|))h(f(A_f))}\right)^{n-1}}. \tag{2.8}$$

Due to (2.4) and (2.8), by the definition of K, we obtain that

$$M\big(\Gamma(f(|K|), f(A_f), \overline{\mathbb{R}^n})\big)\geq \frac{\omega_{n-1}}{\left(\log\frac{\lambda_n^2}{\delta\cdot h(f(x), f(y))}\right)^{n-1}}. \tag{2.9}$$

Since $D_f^{\,\prime}$ is a QED-domain with a constant A_0 in (1.3), under the condition (2.9) we obtain that

$$M\big(\Gamma(f(|K|), f(A_f), D_f^{\,\prime})\big)\geq \frac{\omega_{n-1}}{A_0\left(\log\frac{\lambda_n^2}{\delta\cdot h(f(x), f(y))}\right)^{n-1}}. \tag{2.10}$$

Note that,

$$\Gamma(|K|, A_f, D)>\Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D). \tag{2.11}$$

Indeed, let

$$\gamma\in \Gamma(|K|, A_f, D),\qquad \gamma\colon [0, 1]\rightarrow D,\qquad \gamma(0)\in |K|,\qquad \gamma(1)\in A_f.$$

Since

$$|K|\subset B(x_0, |x-x_0|),\qquad\qquad A_f\subset B(x_0, \varepsilon_0),$$

it follows from (2.7) that $A_f \subset D \setminus B(x_0, \varepsilon_0)$. Then

$$\gamma \cap B(x_0, |x - x_0|) \neq \varnothing \neq \gamma \cap (D \setminus B(x_0, |x - x_0|)).$$

By [12, Theorem 1.I.5.46] there exists

$$t_1 \in (0, 1) \quad \text{such that} \quad \gamma(t_1) \in S(x_0, |x - x_0|).$$

Consider a path $\gamma_1 := \gamma|_{[t_1,1]}$. Since

$$|K| \subset B(x_0, \varepsilon_0), \qquad\qquad A_f \subset D \setminus B(x_0, \varepsilon_0),$$

we get from (2.7) that

$$\gamma_1 \cap B(x_0, \varepsilon_0) \neq \varnothing \neq \gamma \cap (D \setminus B(x_0, \varepsilon_0)).$$

Now by [12, Theorem 1.I.5.46], there exists $t_2 \in (t_1, 1)$ such that

$$\gamma_1(t_2) = \gamma(t_2) \in S(x_0, \varepsilon_0).$$

Let us denote $\gamma_2 := \gamma|_{[t_1,t_2]}$. Then γ_2 is a subpath of γ and

$$\gamma_2 \in \Gamma(S(x_0, |x - x_0|), S(x_0, \varepsilon_0), D).$$

This proves (2.11). In this case, by the minorization principle of the modulus of families of paths (see, e.g., [22, Theorem 6.4]), due to (2.11) and (1.1) we obtain that

$$\begin{aligned} M\big(\Gamma(f(|K|), f(A_f), D_f')\big) &= M\big(f(\Gamma(|K|, A_f, D))\big) \\ &\leq M\big(f(\Gamma(S(x_0, |x - x_0|), S(x_0, \varepsilon_0), D))\big) \\ &\leq \int\limits_{A(x_0, |x-x_0|, \varepsilon_0)} Q(z) \cdot \eta^n(|z - x_0|) dm(z), \end{aligned} \tag{2.12}$$

where η is an arbitrary Lebesgue measurable function satisfying the condition (1.2) for

$$r_1 = |x - x_0|, \qquad r_2 = \varepsilon_0.$$

Since $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$, there is $\sigma > 0$ such that $I(\varepsilon, \varepsilon_0) > 0$ for $0 < \varepsilon < \sigma$. If $|x - x_0| < \sigma$, we put

$$\eta(t) := \begin{cases} \frac{\psi(t)}{I(|x - x_0|, \varepsilon_0)}, & t \in (|x - x_0|, \varepsilon_0); \\ 0, & t \notin (|x - x_0|, \varepsilon_0). \end{cases}$$

Note that the function η satisfies the condition (1.2) or $r_1 = |x - x_0|$ and $r_2 = \varepsilon_0$. Then, due to (2.12) and (2.1) we obtain that

$$M\big(\Gamma(f(|K|), f(A_f), D_f')\big) \leq \tag{2.13}$$

$$\leq \frac{1}{I^n(|x-x_0|, \varepsilon_0)} \int\limits_{A(x_0, |x-x_0|, \varepsilon_0)} \frac{Q(z)}{|z-x_0|^n} dm(z) \tag{2.14}$$

$$\leq \frac{K_0}{I^{n-p}(|x-x_0|, \varepsilon_0)}.$$

Combining (2.10) and (2.14) we have that

$$\frac{\omega_{n-1}}{A_0\left(\log \frac{\lambda_n^2}{\delta \cdot h(f(x), f(y))}\right)^{n-1}} \leq \frac{K_0}{I^{n-p}(|x-x_0|, \varepsilon_0)}.$$

The last relation implies that

$$h(f(x), f(y)) \leq \frac{\lambda_n^2}{\delta} \exp\left\{-\left(\frac{\omega_{n-1}}{A_0 K_0}\right)^{\frac{1}{n-1}} I^{\frac{n-p}{n-1}}(|x-x_0|, \varepsilon_0)\right\}.$$

Observe that the latter relation holds for $\sigma < |x - x_0| < \varepsilon_0$ with some (probably bigger) constant $\frac{1}{C_2} > 0$ instead $\frac{\lambda_n^2}{\delta}$. Indeed, setting

$$C_2 := \exp\left\{-\left(\frac{\omega_{n-1}}{A_0 K_0}\right)^{\frac{1}{n-1}} I^{\frac{n-p}{n-1}}(\sigma, \varepsilon_0)\right\},$$

we obtain that

$$h(f(x), f(y)) \leq 1 = \frac{C_2}{C_2} \leq \frac{1}{C_2} \exp\left\{-\left(\frac{\omega_{n-1}}{A_0 K_0}\right)^{\frac{1}{n-1}} I^{\frac{n-p}{n-1}}(|x-x_0|, \varepsilon_0)\right\}$$

whenever $\sigma < |x - x_0| < \varepsilon_0$. Lemma is proved. □

Proof of Theorem 1.1. Letting $\varepsilon_0 = \delta_0$, where δ_0 is from (1.5), and $\varepsilon < \varepsilon_0$, consider the function $I(\varepsilon, \varepsilon_0) = \int_{\varepsilon}^{\varepsilon_0} \psi(t)dt$ with

$$\psi(t) = \begin{cases} \frac{1}{tq_{x_0}^{\frac{1}{n-1}}(t)}, & t \in (\varepsilon, \varepsilon_0), \\ 0, & t \notin (\varepsilon, \varepsilon_0), \end{cases}$$

(as usual, $a/\infty = 0$ for $a \neq \infty$, $a/0 = \infty$ for $a > 0$ and $0 \cdot \infty = 0$, see [19, Ch. I]). Since f is a homeomorphism, $I(\varepsilon, \varepsilon_0) < \infty$ (see, e.g., [14, Remark 7.2]). In addition, the relation (1.5) implies that $I(\varepsilon, \varepsilon_0) > 0$ for some $\varepsilon_1 \in (0, \varepsilon_0)$ and any $\varepsilon \in (0, \varepsilon_1)$. Let us prove that ψ also satisfies the relation (2.1). Indeed, applying Fubini's theorem (see, e.g., [19, Theorem 8.1.III]) we obtain that

$$\int\limits_{\varepsilon<|z-x_0|<\varepsilon_0} Q(z)\cdot\psi^n(|z-x_0|)dm(z)=\omega_{n-1}\cdot\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}$$
$$=\omega_{n-1}\cdot I(\varepsilon,\varepsilon_0).$$

Consequently, the relation (1.5) implies the relation (2.1) with $p=1$ and $K_0=\omega_{n-1}$. Now, the desired inequality (2.3) follows by Lemma 2.2.

Let us prove that f has a limit as $x\to x_0$. We may to argue by the opposite. Indeed, if f has no a limit as $x\to x_0$, then we may construct at least two sequences $x_m\to x_0$ and $y_m\to x_0$, $m\to\infty$, such that $h(f(x_m),f(y_m))\geq\delta>0$ for some positive $\delta>0$ and all $m=1,2,\ldots$. Let $|x_m-x_0|\geq|y-y_m|$. Then the latter contradicts the inequality (2.3), because the right-hand part of this inequality tends to zero as $m\to\infty$ due to (1.5). Therefore, the limit of the mapping f as $x\to x_0$ exists. □

Proof of Theorem 1.3. Due to [14, Corollary 6.3, Ch. 6], the condition $Q\in FMO(x_0)$ implies that, for some small $\varepsilon<\varepsilon_0<1$,

$$\int\limits_{\varepsilon<|z-x_0|<\varepsilon_0} Q(z)\cdot\psi^n(|z-x_0|)dm(z)=O\left(\log\log\frac{1}{\varepsilon}\right),\quad\varepsilon\to 0,\tag{2.15}$$

where $\psi(t)=\frac{1}{t\log\frac{1}{t}}>0$. Note that the quantity $I(\varepsilon,\varepsilon_0)$ which is defined in (2.2) may be calculated in the following way:

$$I(\varepsilon,\varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)dt=\log\frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon_0}}.$$

Thus, the estimate (2.15) yields

$$\frac{1}{I^n(\varepsilon,\varepsilon_0)}\int\limits_{\varepsilon<|z-x_0|<\varepsilon_0} Q(z)\cdot\psi^n(|z-x_0|)dm(z)\leq C_1\left(\log\log\frac{1}{\varepsilon}\right)^{1-n}.$$

In particular, the condition $I(\varepsilon,\varepsilon_0)\rightarrow\infty$ as $\varepsilon\rightarrow 0$ holds, as well. Now, by Lemma 2.2,

$$h(f(x),f(y))\le\frac{\lambda_n^2}{\delta}\exp\left\{-\left(\frac{\omega_{n-1}}{A_0C_1}\right)^{\frac{1}{n-1}}\log\frac{\log\frac{1}{|x-x_0|}}{\log\frac{1}{\varepsilon_0}}\right\}$$

$$=\frac{\lambda_n^2}{\delta}\left(\frac{\log\frac{1}{\varepsilon_0}}{\log\frac{1}{|x-x_0|}}\right)^{\left(\frac{\omega_{n-1}}{A_0C_1}\right)^{\frac{1}{n-1}}}$$

$$\le C\cdot\left(\frac{1}{\log\frac{1}{|x-x_0|}}\right)^{\left(\frac{\omega_{n-1}}{A_0C_1}\right)^{\frac{1}{n-1}}}.$$

So, the relation (1.6) is proved for $p=\left(\frac{\omega_{n-1}}{A_0C_1}\right)^{\frac{1}{n-1}}$. ☐

In the theory of conformal and quasiconformal mappings, mappings with a fixed point are of particular importance. Based on this, let us consider the following class.

3. MAPPINGS OF DOMAINS WITH COMPLEX BOUNDARIES

In order to prove the statements of Theorems 1.4 and 1.5 concerning domains with complex geometry, we formulate and prove a lemma similar to Lemma 2.2 for this case. The following statement holds.

Lemma 3.1. *Assume that, D is a regular domain and the following conditions are fulfilled:*

1) *for any $y_0\in\partial D$ there is $r_0'=r_0'(y_0)>0$ such that $B(y_0,r)\cap D$ is finitely connected for all $0<r<r_0'$, and, for each connected component K of the set $\overline{B(y_0,r)}\cap D$ the following condition is fulfilled: any pair of points $x,y\in K$ may be joined by a path $\gamma\colon[a,b]\rightarrow\mathbb{R}^n$ such that $|\gamma|\subset K\cap\overline{B(x_0,\max\{|x-y_0|,|y-y_0|\})}$, where*

$$|\gamma|=\{x\in\mathbb{R}^n:\exists t\in[a,b]:\gamma(t)=x\};$$

2) *for any $x_0\in\partial D$, there are $p<n$, $0<\varepsilon_0<\min\{1,r_0'\}$ and a Lebesgue measurable function $\psi\colon(\varepsilon,\varepsilon_0)\rightarrow[0,\infty]$, $\varepsilon\in(0,\varepsilon_0)$ such that the relation*

$$\int\limits_{\varepsilon<|z-x_0|<\varepsilon_0}Q(z)\cdot\psi^n(|z-x_0|)dm(z)\le K_0\cdot I^p(\varepsilon,\varepsilon_0)\qquad(3.1)$$

holds for any $x_0 \in I(P_0)$ *as* $\varepsilon \to 0$ *for some* $K_0 > 0$*, where*

$$0 < I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)dt < \infty$$

for sufficiently small $\varepsilon > 0$*,* $\varepsilon < \varepsilon_0$*.*

Assume that $I(\varepsilon, \varepsilon_0) \to \infty$ *as* $\varepsilon \to 0$*. Then, for any* $P \in E_D$ *there is* $y_0 \in \partial D$ *such that* $I(P) = \{y_0\}$*, where* $I(P)$ *denotes the impression of* P*. Furthermore,* f *has a limit as* $x \to P_0$*, in addition, there exists a neighborhood* U *of* P_0 *in* $\overline{D}_P$ *and a number* $\lambda_n > 0$ *such that the relation*

$$h(f(x), f(y)) \le \frac{\lambda_n^2}{\delta} \exp\left\{-\left(\frac{\omega_{n-1}}{A_0 K_0}\right)^{\frac{1}{n-1}} I^{\frac{n-p}{n-1}}(|x - x_0|, \varepsilon_0)\right\} \qquad (3.2)$$

holds for any $x, y \in U \cap D$*,* $|x - x_0| \ge |y - x_0|$*, and all* $f \in \mathfrak{S}^{A_0}_{Q,a,\delta}(D, P_0)$*, where* $I(P_0) = \{x_0\}$*.*

Proof. **I.** Let $f \in \mathfrak{S}^{A_0}_{Q,a,\delta}(D, P_0)$. Since the set $B(y_0, r) \cap D$ is finitely connected for all $y_0 \in \partial D$ and $0 < r < r_0'(y_0)$, D is finitely connected at the boundary. Therefore, D is uniform domain (see [15, Theorem 3.2]). In other words, for every $r > 0$ there exists a number $\delta > 0$ such that the inequality

$$M\big(\Gamma(F^*, F, D)\big) \ge \delta$$

holds for any continua $F, F^* \subset D$ such that $h(F) \ge r$ and $h(F^*) \ge r$.

II. Let us prove that, for any $P \in E_D$ there exists $y_0 \in \partial D$ such that $I(P) = \{y_0\}$. We prove this statement from the opposite, namely, suppose that there exists a prime end $P \in E_D$, the impression of which contains two points $x, y \in \partial D$, $x \ne y$. In this case, there are at least two sequences $x_m, y_m \in d_m$, $m = 1, 2, \ldots$, which converge to x and y as $m \to \infty$, respectively (here d_m denotes the decreasing sequence of domains formed by the corresponding sequence of cuts σ_m in P).

Let us join the points x_m and d_m with a path γ_m in d_m. Since $x \ne y$, there exists $m_0 \in \mathbb{N}$ such that $h(|\gamma_m|) \ge d(x, y)/2$, $m > m_0$. Choose any nondegenerate continuum $C \subset D \setminus d_1$. Then, due to the uniformity of the domain D

$$M\big(\Gamma(|\gamma_m|, C, D)\big) \ge \delta_0 > 0 \qquad (3.3)$$

for some $\delta_0 > 0$ and all $m > m_0$. The inequality (3.3) contradicts the definition of a prime end P. Indeed, by the definition of σ_m we have that $\Gamma(|\gamma_m|, C, D) > \Gamma(\sigma_m, C, D)$. Now, due to (1.7) we obtain that

$$M\big(\Gamma(|\gamma_m|, C, D)\big) \le M\big(\Gamma(\sigma_m, C, D)\big) \to 0$$

as $m \to \infty$. The last relation contradicts (3.3), so $I(P) = \{y_0\}$ for some $y_0 \in \partial D$.

III. It remains to prove the relation (3.2). Let $I(P_0) = \{x_0\} \in \partial D$. Let d_m be a sequence of domains corresponding to cuts σ_m in P_0 such that $\sigma_m \subset S(x_0, r_m)$, $x_0 \in \partial D$ and $r_m \to 0$ as $m \to \infty$ (see [11, Lemma 2]).

Since D is regular, the space $\overline{D}_P$ contains no less than two prime ends P_1 and $P_2 \in E_D$. Let $P_1 \subset E_D$ is a prime end which does not coincide with P_0. Suppose that, G_m, $m = 1, 2, \ldots$, is a sequence of domains that corresponds to P_1. Let us construct a sequence of continua K_m, $m = 1, 2, \ldots$, as follows. Join the points ζ_1 and a by an arbitrary path in D, which we denote by K_1. Further, join the points ζ_2 and ζ_1 by the path K_1' in G_1.

By combining paths K_1 and K_1', we obtain a path K_2, joining points a and ζ_2. And so on. Let K_m be a path joining the points ζ_m and a. Let us join the points ζ_{m+1} and ζ_m by a path K_m', which lies in G_m. By uniting the paths K_m and K_m', we obtain a path K_{m+1}. And so on.

Let us show that, there is a number $m_1 \in \mathbb{N}$ such that

$$d_m \cap K_m = \varnothing \quad \forall \quad m \geq m_1. \tag{3.4}$$

Let us prove it from the opposite, namely, suppose that (3.4) is not true. Then, there is an increasing sequence of numbers $m_k \to \infty$, $k \to \infty$, and points $\xi_k \in K_{m_k} \cap d_{m_k}$, $m = 1, 2, \ldots,$. Then $\xi_k \to P_0$ as $k \to \infty$.

Note that, there are possible two cases: either all ξ_k belong to $D \setminus G_1$ for $k = 1, 2, \ldots$, or there is $k_1 \in \mathbb{N}$ such that $\xi_{k_1} \in G_1$. In the second case, consider a sequence ξ_k, $k > k_1$. Note that, two cases are possible: either ξ_k for $k > k_1$ belong to $D \setminus G_2$, or there is $k_2 > k_1$ such that $\xi_{k_2} \in G_2$. In the second case, consider a sequence ξ_k, $k > k_2$, and so on.

Suppose that the element $\xi_{k_{l-1}} \in G_{l-1}$ is already constructed. Note that, there are two cases: either ξ_k belong to $D \setminus G_l$ for $k > k_{l-1}$, or there is a number $k_l > k_{l-1}$ such that $\xi_{k_l} \in G_l$, and etc. This procedure may be both finite and infinite, depending on which we have two possible situations:

1) there are $n_0 \in \mathbb{N}$ and $l_0 \in \mathbb{N}$ such that $\xi_k \in D \setminus G_{n_0}$ for all $k > l_0$;
2) for each $l \in \mathbb{N}$ there is ξ_{k_l} such that $\xi_{k_l} \in G_l$ and the sequence k_l is increasing in $l \in \mathbb{N}$.

Consider each of these cases separately and show that we come to a contradiction in both of them. Suppose that the case 1) holds. Observe that, all elements of the sequence ξ_k belong to K_{n_0}, which implies the existence of the subsequence ξ_{k_r}, $r = 1, 2, \ldots$, which converges as $r \to \infty$ to some point $\xi_0 \in D$. However, $\xi_k \in d_{m_k}$ and, therefore (see [11, Proposition 1])

$$\xi_0\in\bigcap\limits_{m=1}^{\infty}\overline{d_m}\subset\partial D.$$

The resulting contradiction indicates the impossibility of case 1).

Consider now the case 2). Then simultaneously $\xi_k\rightarrow P_0$ and $\xi_k\rightarrow P_1$ as $k\rightarrow\infty$. Since the space $\overline{D}_P$ is metric (see [11, Remark 2], by the triangle inequality, it follows that $P_1=P_0$, which contradicts the choice of P_1. The resulting contradiction indicates the validity of the relation (3.4).

IV. As proved above, $I(P_1)=\{z_0\}$, $z_0\in\partial D$. Therefore, by construction, there exists $z_m\rightarrow z_0$, $m\rightarrow\infty$, where $x_m\in K$, and K is a path with the origin at the point a, which comes from the combination of K_m, $m=1,2,\ldots$, constructed above. Recall that

$$C(y_0,f)\subset\partial D_f^{\,\prime}\,,\qquad\qquad D_f^{\,\prime}:=f(D),$$

where $C(y_0,f)$ is the cluster set of f at the point y_0 (see, e.g., [14, Proposition 13.5]). Reasoning in the same way as when proving Theorem 1.1, it is possible to prove the existence of a point $a_f\in K$, such that

$$h(f(a),f(a_f))\geq(1/2)\cdot h(f(a),\partial D_f^{\,\prime})\geq\delta/2. \tag{3.5}$$

Now let A_f be a continuum in D, which is part of the path E from point a to a point a_f.

V. Since $I(P_0)=\{x_0\}$, we may find a neighborhood U of P_0 in $\overline{D}_P$ such that

$$U\cap D\subset B(x_0,\varepsilon_0).$$

By the definition of a regular domain, we may consider that $U\cap D$ is connected. Now $U\cap D$ belongs to one and only one component K of $B(x_0,\varepsilon_0)\cap D$. Let $x,y\in U\cap D$ such that $|x-x_0|\geq|y-x_0|$. By the definition of $r_0^{\prime}$, points x and y may be joined by a path K, contained in the ball $\overline{B(x_0,|x-x_0|)}$.

In what follows, the proof is very similar to the proof of Theorem 1.1. We may consider that, K and A_f are Jordan paths.

Note also that, the paths $f(K)$ and $f(A_f)$ are also Jordan and do not divide the space $\mathbb{R}^n$ (see proof Theorem 1.1). Therefore,

$$R=R(f(|K|),f(A_f))$$

is a ring domain. Denote $\Gamma=\Gamma(f(|K|),f(A_f),\mathbb{R}^n)$. Then, by Proposition 2.1

$$M\big(\Gamma(f(|K|),f(A_f),\overline{\mathbb{R}^n})\big)\geq\frac{\omega_{n-1}}{\Big(\log\frac{2\lambda_n^2}{h(f(|K|))h(A_f)}\Big)^{n-1}}. \tag{3.6}$$

Due to (3.5) and (3.6), by definition of A_f,

$$M\big(\Gamma(f(|K|), f(A_f), \overline{\mathbb{R}^n})\big) \geq \frac{\omega_{n-1}}{\left(\log \frac{\lambda_n^2}{\delta\cdot h(f(x),f(y))}\right)^{n-1}}. \tag{3.7}$$

Since D_f' is a QED-domain with constant A_0 in (1.3), under the condition (3.7) we obtain that

$$M\big(\Gamma(f(|K|), f(f(A_f)), D_f')\big) \geq \frac{\omega_{n-1}}{A_0\left(\log \frac{\lambda_n^2}{\delta\cdot h(f(x),f(y))}\right)^{n-1}}. \tag{3.8}$$

Note that

$$\Gamma(|K|, A_f, D) > \Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D). \tag{3.9}$$

The relation (3.9) may be proved in the same way as (2.11), which was established under the proof of Theorem 1.1. In this case, by minorization of the modulus (see, e.g., [22, Theorem 6.4]), due to (2.11) and (1.1) we obtain that

$$\begin{aligned} M\big(\Gamma(f(|K|), f(A_f), D_f')\big) &= M\big(f(\Gamma(|K|, A_f, D))\big) \\ &\leq M\big(f(\Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D))\big) \\ &\leq \int\limits_{A(x_0,|x-x_0|,\varepsilon_0)} Q(z)\cdot \eta^n(|z-x_0|)dm(z), \end{aligned}$$

where η is an arbitrary Lebesgue measurable function satisfying the relation (1.2) for $r_1 = |x-x_0|$, $r_2 = \varepsilon_0$. Since $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$, there is $\sigma > 0$ such that $I(\varepsilon, \varepsilon_0) > 0$ for $0 < \varepsilon < \sigma$. If $|x-x_0| < \sigma$, we put

$$\eta(t) := \begin{cases} \frac{\psi(t)}{I(|x-x_0|,\varepsilon_0)}, & t \in (|x-x_0|, \varepsilon_0), \\ 0, & t \notin (|x-x_0|, \varepsilon_0). \end{cases}$$

Note that the function η satisfies the condition (1.2) for $r_1 = |x-x_0|$ and $r_2 = \varepsilon_0$. Then, due to (3.9) and (3.1) we obtain that

$$\begin{aligned} M\big(\Gamma(f(|K|), f(A_f), D_f')\big) &\leq \\ &\leq \frac{1}{I^n(|x-x_0|, \varepsilon_0)} \int\limits_{A(x_0,|x-x_0|,\varepsilon_0)} \frac{Q(z)}{|z-x_0|^n} dm(z) \\ &\leq \frac{K_0}{I^{n-p}(|x-x_0|, \varepsilon_0)}. \end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10) we have that

$$\frac{\omega_{n-1}}{A_0\left(\log \frac{\lambda_n^2}{\delta\cdot h(f(x),f(y))}\right)^{n-1}} \leq \frac{K_0}{I^{n-p}(|x-x_0|, \varepsilon_0)}.$$

The latter relation implies that

$$h(f(x),f(y))\leq\frac{\lambda_n^2}{\delta}\exp\left\{-\left(\frac{\omega_{n-1}}{A_0K_0}\right)^{\frac{1}{n-1}}I^{\frac{n-p}{n-1}}(|x-x_0|,\varepsilon_0)\right\}.$$

Arguing similarly to the last part of the proof of Lemma 2.2, we may prove that the latter relation holds for $\sigma<|x-x_0|<\varepsilon_0$ with some (probably bigger) constant $\frac{1}{C_2}>0$ instead $\frac{\lambda_n^2}{\delta}$. Lemma is proved. $\Box$

Proof of Theorems 1.4 and 1.5 follow from Lemma 3.1 similarly to the proof of Theorems 1.1 and 1.3.

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