

# On equicontinuity of mappings with branching defined in family of domains

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All definitions and notions used below may be found in [1]. Let  $S_i = S(x_0, r_i) = \{x \in \mathbb{R}^n : |x - x_0| = r_i\}$ ,  $i = 1, 2$ ,  $n \geq 2$ , and let  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$  be a Lebesgue measurable function satisfying the condition  $Q(x) \equiv 0$  for  $x \in \mathbb{R}^n \setminus D$ . In what follows,  $M_p(\cdot)$  denotes the modulus of family of paths of the order  $p \geq 1$ , and  $dm(x)$  is an element of the Lebesgue measure in  $\mathbb{R}^n$ . Let  $A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$ . Given  $p \geq 1$ , a mapping  $f : D \rightarrow \overline{\mathbb{R}^n}$  is called a *ring  $Q$ -mapping at a point  $x_0 \in \mathbb{R}^n$  with respect to  $p$ -modulus*, if the condition

$$M_p(f(\Gamma(S_1, S_2, D))) \leq \int_{A \cap D} Q(x) \cdot \eta^p(|x - x_0|) dm(x) \quad (1)$$

holds for all  $0 < r_1 < r_2 < \infty$  and any Lebesgue measurable function  $\eta : (r_1, r_2) \rightarrow [0, \infty]$  such that

$$\int_{r_1}^{r_2} \eta(r) dr \geq 1. \quad (2)$$

Below  $h(x, y)$  denotes the chordal (spherical metric) between points  $x, y \in \overline{\mathbb{R}^n}$ . Let  $h(E)$  be achordal diameter of the set  $E$  in  $\overline{\mathbb{R}^n}$ . For  $x_0 \in \mathbb{R}^n$  and  $r_0 > 0$ , as usual,  $B(x_0, r_0) = \{x \in \mathbb{R}^n : |x - x_0| < r_0\}$ . Given  $p \geq 1$ , a domain  $D_0 \subset \mathbb{R}^n$ , a set  $E \subset \overline{\mathbb{R}^n}$  and a number  $\delta > 0$ , a fixed domain  $D_0 \subset \mathbb{R}^n$ ,  $n \geq 2$ , a sequence of domains  $\mathfrak{D} = \{D_m\}_{m=1}^{\infty}$  the kernel of which is  $D_0$  and a Lebesgue measurable function  $Q : \mathbb{R}^n \rightarrow [0, \infty]$  we denote by  $\mathfrak{R}_{Q, \delta, p, E}(D_0, \mathfrak{D})$  the family of all open discrete closed mappings  $f : D_m \rightarrow \overline{\mathbb{R}^n} \setminus E$  satisfying (1)–(2) for any  $0 < r_1 < r_2 < \infty$  at any point  $x_0 \in \overline{D_0}$  such that the following condition holds: for any domain  $D'_m := f_m(D_m)$  there exists a continuum  $K_m \subset D'_m$  such that  $h(K_m) \geq \delta$  and  $h(f_m^{-1}(K_m), \partial D_m) \geq \delta > 0$ . The following statement holds.

**Theorem 1.** *Let  $p \in (n - 1, n]$  and let  $f_m \in \mathfrak{R}_{Q, \delta, p, E}(D_0, \mathfrak{D})$ ,  $m = 1, 2, \dots$ , be a sequence such that:*

- 1) *the sequence of domains  $D_m$  is regular with respect to  $D_0$ ;*
- 2) *for every  $m \in \mathbb{N}$ , a domain  $D_m$  is locally connected on its boundary;*
- 3) *the family  $f_m(D_m)$  is equi-uniform with respect to  $p$ -modulus over all  $m \in \mathbb{N}$ ;*
- 4) *at least one of two following conditions hold:  $Q$  has a finite mean oscillation in  $\overline{D_0}$ , or*

$$\int_0^{\beta(x_0)} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)} = \infty \quad (3)$$

for every  $x_0 \in \overline{D_0}$  and some  $\beta(x_0) > 0$ , where  $q'_{x_0}(t)$  denotes the integral average of the function  $Q'$  under the sphere  $S(x_0, t)$ , while  $Q'(x) = \begin{cases} Q(x), & Q(x) \geq 1, \\ 1, & Q(x) < 1. \end{cases}$  Let  $E$  be a set of a positive capacity for  $p = n$ , and  $E$  is arbitrary closed set for  $n - 1 < p < n$ .

**I.** Then the family  $f_m$ ,  $m = 1, 2, \dots$ , is uniformly equicontinuous in  $\mathfrak{D}$ , i.e., for any  $\varepsilon > 0$  there is  $\delta = \delta(\varepsilon) > 0$  such that  $h(f_m(x), f_m(y)) < \varepsilon$  whenever  $x, y \in D_m$ ,  $|x - y| < \delta$  and  $m \in \mathbb{N}$ . Moreover, there is a subsequence  $f_{m_k}$ ,  $k = 1, 2, \dots$ , which converges to  $f$  locally uniformly in  $D_0$ . In this case,  $f$  has a continuous boundary extension  $f : \overline{D_0} \rightarrow \overline{\mathbb{R}^n}$ . If  $x_m \in D_m$ ,  $m = 1, 2, \dots$ ,  $f_m$  converges to  $f$  locally uniformly in  $D_0$  and  $x_m \rightarrow x_0$  as  $m \rightarrow \infty$ , then  $f_m(x_m) \rightarrow f(x_0)$ .

**II.** In addition, assume that

5)  $p = n$  and there is  $r > 0$ , does not depending on  $m \in \mathbb{N}$ , such that  $h(E) \geq r$  whenever  $E$  is any component of  $\partial f_m(D_m)$ . If  $A_0$  is some compactum in  $D_0$ , then there is  $\delta_2 > 0$  and  $M_0 \in \mathbb{N}$  such that  $A_0 \subset D_m$  and  $h(f_m(A_0), \partial f_m(D_m)) \geq \delta_2$  for every  $m \geq M_0$ . The mapping  $f$  is boundary preserving: if  $x_0 \in \partial D_0$ , then  $f(x_0) \in \partial f(D_0)$ . If  $\overline{B(x_0, \varepsilon_0)} \subset D_0 \cap \bigcap_{m=1}^{\infty} D_m$ , then there is  $\varepsilon_1 > 0$  does not depending on  $m \in \mathbb{N}$  such that

$$B_h(f_m(x_0), \varepsilon_1) \subset f_m(D_m), \quad m = 1, 2, \dots, \quad (4)$$

where  $B_h(y_0, r_0) = \{y \in \overline{\mathbb{R}^n} : h(y, y_0) < r_0\}$ .

The mentioned result is obtained in [2].

#### REFERENCES

- [1] Martio O., Ryazanov V., Srebro U. and Yakubov E. *Moduli in Modern Mapping Theory*. Springer Science + Business Media, LLC : New York, 2009.
- [2] Ilkevych N., Romash D., Sevost'yanov E. On distortion estimates under mappings of families of domains. <https://arxiv.org/abs/2602.14013> .

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