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ON SINGULARITIES OF MAPPINGS WITH A MAJORANT INTEGRABLE ON SPHERES

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The paper is devoted to the study of the boundary behavior of mappings with finite distortion, more precisely, open discrete mappings with moduli conditions similar to Poletsky inequality in the inverse direction. We study the case when some majorant in the above inequality is integrable over spheres centered at each finite point. It is established that the indicated mappings have a continuous extension to an isolated boundary point of the boundary of a domain whenever the mapping f omits at least one a point. The proof of the main results is step-by-step and is based on the following logic, in line with which we prove that 1) no two different boundary points of the corresponding mapped domain can belong to the limit set of the mapping f at the point x_0 , 2) the cluster set of a mapping at a given point is, in principle, always a singleton, provided that this mapping omits at least one finite point. The proofs of the these statements are made by the contradiction. This contradiction, in turn, is ensured by the property of approaching continua in the preimage under the mapping on the one hand, and by the upper bound on the mapped families of paths (taking into account the definition of the mapping through this upper bound) on the other hand. In order to construct the above approaching continua, we use some geometric constructions that take into account the nature (definition) of the class of mappings under study. In particular, to prove that the cluster set under the mapping does not contain more than one boundary point, we use segments joining elements of some approximate sequences with its limit points, and the corresponding converging continua which are the pre-images of these segments under the mapping. Note that, the result obtained in the article was previously obtained by us first for homeomorphisms, then for open discrete closed mappings, and then for open discrete mappings but with an integrable majorant in the inverse Poletsky inequality. We also previously established a similar result, but in the case where the mapping omits at least two points.

1. Introduction. This paper is devoted to the problem of removability of isolated singularities of mappings with finite distortion. We should note that a significant number of papers on this topic have been devoted to this issue; see, for example, [1–2], [3], [4], [5] and [6]. In particular, we study this problem in the context of mappings with inverse moduli inequalities. Several of our results on this topic have been published recently, see [7], [8], [9] and [10]. Note that in [7], this problem was studied for homeomorphisms, and in [8], for boundary-preserving open discrete mappings (i.e., mappings whose corresponding cluster set lies on the boundary of the mapped domain). In our last two papers, [9] and [10], we went much further in this matter and abandoned the boundary-preserving requirement. In particular, we found

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that open discrete mappings with an integrable majorant may be continuously extended to an isolated point of the boundary. Moreover, if a mapping omits a pair of values, then integrability of the corresponding majorant over almost all spheres is sufficient (see [10]). The problem mentioned above has not yet been fully solved. For example, it is not entirely clear to what extent the omitting of two or more points under the mapping is necessary. Moreover, the problem of essentially singular points has not been fully investigated, because in the mentioned papers there is no description of mappings with such properties. At the same time, the corresponding problem of mappings with a “direct” inequality was solved by us much earlier, even in the situation of general metric spaces ([11]).

This manuscript is devoted to the study of a case similar to that considered in [9]. Specifically, we study the problem in a domain of the Euclidean space where the omitting of one point is assumed. At the same time, we consider the case where the corresponding function Q is integrable only on almost all spheres at finite points of the space. Under this condition, we prove a continuous boundary extension of an open discrete mapping to an isolated point on the boundary.

Let us recall some definitions. Let us denote an n -dimensional Euclidean space \mathbb{R}^n with its standard metric and topology. Everywhere below, the boundary and the closure of the set $A \subset \mathbb{R}^n$ are denoted by ∂A and \overline{A} , respectively, and they should be understood in the sense of the Alexandroff one-point compactification $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$ with the *spherical (chordal) metric* $h(x, y) = |\pi(x) - \pi(y)|$, where

$$h(x, \infty) = \frac{1}{(1 + |x|^2)^{1/2}}, \quad h(x, y) = \frac{|x - y|}{(1 + |x|^2)^{1/2}(1 + |y|^2)^{1/2}}, \quad x \neq \infty \neq y, \quad (1)$$

(see, e.g., [6, Definition 12.1]).

Also, everywhere below D is a domain in \mathbb{R}^n , $n \geq 2$. A Borel function $\rho: \mathbb{R}^n \rightarrow [0, \infty]$ is called *admissible* for the family Γ of paths γ in \mathbb{R}^n , if the relation

$$\int_{\gamma} \rho(x) |dx| \geq 1$$

holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \text{adm } \Gamma$. Let $p \geq 1$, then p -*modulus* of Γ is defined by the equality

$$M_p(\Gamma) = \inf_{\rho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \rho^p(x) dm(x).$$

We set $M(\Gamma) := M_n(\Gamma)$. Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A(y_0, r_1, r_2) = \{y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2\}. \quad (2)$$

As usually, for $x_0 \in \mathbb{R}^n$ and $r > 0$ we set

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \quad B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}.$$

Given sets $E, F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma: [a, b] \rightarrow \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. If $f: D \rightarrow \mathbb{R}^n$ is a given mapping, $y_0 \in \overline{f(D)}$ and $0 < r_1 < r_2 < d_0 = \sup_{y \in f(D)} |y - y_0|$, then by $\Gamma_f(y_0, r_1, r_2)$ we

denote the family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q: \mathbb{R}^n \rightarrow [0, \infty]$ is a Lebesgue measurable function. We say that f *satisfies inverse Poletsky inequality relative to p -modulus* at the point $y_0 \in \overline{f(D)}$, $p \geq 1$, if there is $r_0 = r_0(y_0) > 0$, $r_0 < d_0 = \sup_{y \in f(D)} |y - y_0|$, such that the ratio

$$M_p(\Gamma_f(y_0, r_1, r_2)) \leq \int_{f(D) \cap A(y_0, r_1, r_2)} Q(y) \cdot \eta^p(|y - y_0|) dm(y) \quad (3)$$

holds for any $0 < r_1 < r_2 < r_0$ and any Lebesgue measurable function $\eta: (r_1, r_2) \rightarrow [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) dr \geq 1. \quad (4)$$

Set $C(x, f) := \{y \in \overline{\mathbb{R}^n} : \exists x_k \in D : x_k \rightarrow x, f(x_k) \rightarrow y, k \rightarrow \infty\}$.

A mapping $f: D \rightarrow \mathbb{R}^n$ is called *discrete* if the preimage $f^{-1}(y) = \{x \in \mathbb{R}^n : f(x) = y\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in \mathbb{R}^n . Further, for the sets $A, B \subset \overline{\mathbb{R}^n}$ we put

$$h(A, B) = \inf_{x \in A, y \in B} h(x, y), \quad h(A) = \sup_{x, y \in A} h(x, y). \quad (5)$$

Similarly, for the sets $A, B \subset \mathbb{R}^n$ we put

$$d(A, B) = \inf_{x \in A, y \in B} |x - y|, \quad d(A) = \sup_{x, y \in A} |x - y|.$$

The following theorem indicates that there cannot be more than one limit point among the boundary points of the mapping f at the point x_0 .

Theorem 1. *Let D be a domain in \mathbb{R}^n , $n \geq 2$, let $p \geq n$, $x_0 \in D$ and let $f: D \setminus \{x_0\} \rightarrow \mathbb{R}^n$ be an open discrete mapping that satisfies conditions (3)–(4) at least at one finite point $y_0 \in C(x_0, f) \cap \partial D'$, $D' := f(D \setminus \{x_0\})$. Assume that, there is $r_0 = r_0(y_0) > 0$, $r_0 < d_0 = \sup_{y \in f(D)} |y - y_0|$, such that for any $0 < r_1 < r_2 < r_0$ there is a set $E \subset [r_1, r_2]$ of positive linear Lebesgue measure such that the function Q is integrable on $S(y_0, r)$ for any $r \in E$ relative to the $(n - 1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1} on $S(y_0, r)$. Then the set $C(x_0, f) \cap \partial D'$ cannot contain more than one point.*

Let $N(y, f, A) = \text{card } \{x \in A : f(x) = y\}$, $N(f, A) = \sup_{y \in \mathbb{R}^n} N(y, f, A)$.

The following statement is true, cf. [9] and [10].

Theorem 2. *Let $n \geq 2$, $p \geq n$, let D be a domain in \mathbb{R}^n , let $x_0 \in D$, and let $f: D \setminus \{x_0\} \rightarrow D'$, $D' := f(D \setminus \{x_0\}) \subset \mathbb{R}^n$, be an open discrete mapping that satisfies conditions (3)–(4) at any point $y_0 \in \overline{D'} \setminus \{\infty\}$.*

Assume that, in addition, for any $y_0 \in \overline{D'} \setminus \{\infty\}$ there is $r_0 = r_0(y_0) > 0$, $r_0 < d_0 = \sup_{y \in f(D)} |y - y_0|$, such that, for any $0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|$ there exists a set $E \subset [r_1, r_2]$ of positive linear Lebesgue measure such that the function Q is integrable on $S(y_0, r)$ relative to the $(n - 1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1} on $S(y_0, r)$ for any $r \in E$.

Assume also that, one of the following conditions hold: (1) $C(x_0, f) \neq \overline{\mathbb{R}^n}$, or (2) $D' \neq \mathbb{R}^n$, or (3) $N(y, f, D \setminus \{x_0\}) < \infty$ at least for one $y \in \mathbb{R}^n$.

Then f has a continuous open and discrete extension $\bar{f}: D \rightarrow \overline{\mathbb{R}^n}$.

Corollary 1. *The statement of Theorem 2 holds if the condition on Q is replaced by a simpler one: $Q \in L^1_{\text{loc}}(\overline{D'} \setminus \{\infty\})$.*

Remark 1. Due to [10, Theorem 1], the following statement is true:

Let $n \geq 2$, let D be a domain in \mathbb{R}^n , $x_0 \in D$, $a, b \in \mathbb{R}^n$, $a \neq b$ and let $f: D \setminus \{x_0\} \rightarrow \mathbb{R}^n \setminus \{a, b\}$ be an open discrete mapping satisfying (3)–(4) for any $y_0 \in \overline{D'}$, where $D' := f(D \setminus \{x_0\})$. Assume that, for any $y_0 \in \overline{D'}$ there is $r_0 = r_0(y_0) > 0$ such that $q_{y_0}(r) < \infty$ for a.a. $r \in (0, r_0)$. Then f has a continuous extension $\bar{f}: D \rightarrow \overline{\mathbb{R}^n}$, the continuous of which is understood in the sense of the chordal metrics h . Moreover, \bar{f} is open and discrete in D .

We should clarify that the result of Theorem 2, item (2), do not follow from this result, because in Theorem 2 we assume that the mapping omits only one point, and in the result mentioned in Remark 1 we mean at least two points a and b .

2. Auxiliary statements. Before starting the proof of the main ones results, we give the following statements.

Proposition 1. ([9, Proposition 2.1]). *Let D be a domain in \mathbb{R}^n , $n \geq 2$, $x_0 \in D$, and let $f: D \setminus \{x_0\} \rightarrow \mathbb{R}^n$, $n \geq 2$, be an arbitrary mapping. Then $C(x_0, f)$ is a continuum in $\overline{\mathbb{R}^n}$.*

The following statement can be found in [12, Theorem 1.I.5.46].

Proposition 2. *Let A be a set in a topological space X . If C is connected and $C \cap A \neq \emptyset \neq C \setminus A$, then $C \cap \partial A \neq \emptyset$.*

Let $D \subset \mathbb{R}^n$, $f: D \rightarrow \mathbb{R}^n$ be a discrete open mapping, $\beta: [a, b) \rightarrow \mathbb{R}^n$ be a path, and $x \in f^{-1}(\beta(a))$. A path $\alpha: [a, c) \rightarrow D$ is called a *maximal f -lifting* of β starting at x , if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a, c)}$; (3) for $c < c' \leq b$, there is no a path $\alpha': [a, c') \rightarrow D$ such that $\alpha = \alpha'|_{[a, c)}$ and $f \circ \alpha' = \beta|_{[a, c')}$. If $\beta: [a, b) \rightarrow \overline{\mathbb{R}^n}$ is a path and if $C \subset \overline{\mathbb{R}^n}$, we say that $\beta \rightarrow C$ as $t \rightarrow b$, if the spherical distance $h(\beta(t), C) \rightarrow 0$ as $t \rightarrow b$ (see [14, section 3.11]), where $h(\beta(t), C)$ is defined in [5]. The following assertion holds (see [14, Lemma 3.12]).

Proposition 3. *Let $f: D \rightarrow \mathbb{R}^n$, $n \geq 2$, be an open discrete mapping, let $x_0 \in D$, and let $\beta: [a, b) \rightarrow \mathbb{R}^n$ be a path such that $\beta(a) = f(x_0)$ and such that either $\lim_{t \rightarrow b} \beta(t)$ exists, or $\beta(t) \rightarrow \partial f(D)$ as $t \rightarrow b$. Then β has a maximal f -lifting $\alpha: [a, c) \rightarrow D$ starting at x_0 . If $\alpha(t) \rightarrow x_1 \in D$ as $t \rightarrow c$, then $c = b$ and $f(x_1) = \lim_{t \rightarrow b} \beta(t)$. Otherwise $\alpha(t) \rightarrow \partial D$ as $t \rightarrow c$.*

The following result is true, see e.g. [9, Proposition 2.4].

Proposition 4. *Let $n \geq 2$, D be a domain in \mathbb{R}^n , $x_0 \in D$, and let $f: D \setminus \{x_0\} \rightarrow \mathbb{R}^n$ be an open discrete mapping. If f has a continuous extension $\bar{f}: D \rightarrow \overline{\mathbb{R}^n}$ to a point x_0 , then \bar{f} is also an open and discrete mapping.*

The following statement contains an elementary connection between the integrability of a function on spheres and its integrability by the Lebesgue measure. Note that this statement is only one of the versions of Fubini's theorem (see, e.g., [15, Theorem 8.1.III]).

Proposition 5. *Let D be a domain in \mathbb{R}^n , $n \geq 2$, $Q \in L^1(D')$, $Q \equiv 0$ on $\mathbb{R}^n \setminus D'$. Then for every $y_0 \in \overline{D'} \setminus \{\infty\}$ and each $0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|$ the function Q is integrable on $S(y_0, r)$ for almost any $r_1 \leq r \leq r_2$ with respect to the $(n - 1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1} on $S(y_0, r)$.*

Proof. It follows from the conditions of Proposition 5 that $Q \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then by Fubini's theorem (see, e.g., [15, Theorem 8.1.III]) we obtain that

$$\int_{r_1 < |y - y_0| < r_2} Q(y) dm(y) = \int_{r_1}^{r_2} \int_{S(y_0, r)} Q(y) d\mathcal{H}^{n-1}(y) dr < \infty.$$

This means that Q is integrable on $S(y_0, r)$ for almost all $r \in [r_1, r_2]$, as required. \square

The following statement holds, see [16, Lemma 2.2], cf. [6, Theorem 10.12].

Lemma 1. *Let D be a domain in \mathbb{R}^n , $n \geq 2$, and $x_0 \in D$. Then, for any $P > 0$ and any neighborhood U of the point x_0 there is a neighborhood $V \subset U$ of the same point, such that the inequality $M(\Gamma(E, F, D)) > P$ holds for any continua $E, F \subset D$ which intersect ∂U and ∂V .*

Remark 2. Since the modulus of the family of paths passing through a fixed point is zero (see section 7.9 in [6]), the statement of Lemma 1 remains true for the case when x_0 is an isolated point of the boundary of the domain D .

3. Proof of Theorem 1. Without loss of generality, we may consider that D is a bounded domain. We will partially use and improve the approach, performed in the proof of Theorem 1 in [8], cf. Theorem 6 in [13]. Let us prove this assertion by contradiction. Let $z_1, z_2 \in C(x_0, f) \cap \partial D'$, $z_1 \neq z_2$. We may assume that $z_1 \neq \infty$. Then we may also find some sequences $x_m, x'_m \in D \setminus \{x_0\}$, $m = 1, 2, \dots$, such that $x_m, x'_m \rightarrow x_0$ as $m \rightarrow \infty$, and $y_m := f(x_m) \rightarrow z_1$ and $y'_m := f(x'_m) \rightarrow z_2$ as $m \rightarrow \infty$. Now, we may find some numbers $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\overline{B(z_1, \varepsilon_0)} \cap \overline{B_*(z_2, \varepsilon_1)} = \emptyset,$$

where $B_*(z_2, \varepsilon_1) = B(z_2, \varepsilon_1)$ for $z_2 \neq \infty$ and $B_*(z_2, \varepsilon_1) = \{y \in \overline{\mathbb{R}^n} : h(y, \infty) < \varepsilon_1\}$ for $z_2 = \infty$. We may consider that $0 < \varepsilon_0 < r_0$, where r_0 is a number from the conditions of the theorem. Note that, $B(z_1, \varepsilon_0)$ is a convex set, and $B_*(z_2, \varepsilon_1)$ is path connected. In this case, points y_m and z_1 may be joined by the segment $I_m(t) = y_m + t(z_1 - y_m)$, $t \in (0, 1)$, which completely belongs to $B(z_1, \varepsilon_0)$. Similarly, points y'_m and z_2 may be joined by a path $J_m(t)$, $t \in [0, 1]$, which belongs to the "ball" $B_*(z_2, \varepsilon_1)$. In what follows, given a path $\gamma : [a, b] \rightarrow \mathbb{R}^n$ we set

$$|\gamma| = \{x \in \mathbb{R}^n : \exists t \in [a, b] : \gamma(t) = x\}.$$

By the construction, $|I_m| \cap \partial D' \neq \emptyset \neq |J_m| \cap \partial D'$. Set

$$t_m := \sup_{t \in [0, 1] : I_m(t) \in D'} t, \quad p_m := \sup_{t \in [0, 1] : J_m(t) \in D'} t, \quad C_1^m := I_m|_{[0, t_m]}, \quad C_2^m := J_m|_{[0, p_m]}.$$

By Proposition 3, the paths C_1^m and C_2^m have maximal f -liftings $C_1^{m*} : [0, c_1^m) \rightarrow D \setminus \{x_0\}$ and $C_2^{m*} : [0, c_2^m) \rightarrow D \setminus \{x_0\}$ at the starting at the points x_m and x'_m , respectively. Note that, the case $C_1^{m*}(t) \rightarrow z_0$ as $t \rightarrow c_1^m - 0$, where $z_0 \in D \setminus \{x_0\}$, is impossible, because in this situation, by Proposition 3 we would have that $c_1^m = t_m$ and $I_m(t) \rightarrow f(z_0) \in D'$, which contradicts the definition of t_m . Now, again by Proposition 3

$$h(C_1^{m*}(t), \partial(D \setminus \{x_0\})) \rightarrow 0, \quad t \rightarrow c_1^m - 0. \quad (6)$$

Similarly, it may be shown that

$$h(C_2^{m*}(t), \partial(D \setminus \{x_0\})) \rightarrow 0, \quad t \rightarrow c_2^m - 0. \quad (7)$$

By (6) and (7), the following four cases are possible:

- 1) $C_1^{m*}(t) \rightarrow \partial D$ as $t \rightarrow c_1^m - 0$ and $C_2^{m*}(t) \rightarrow \partial D$ as $t \rightarrow c_2^m - 0$ for every $m \in \mathbb{N}$;
- 2) $C_1^{m*}(t) \rightarrow \partial D$ as $t \rightarrow c_1^m - 0$, but there is $m_0 \in \mathbb{N}$ such that $C_2^{m_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_2^{m_0} - 0$;
- 3) there is $m_0 \in \mathbb{N}$ such that $C_1^{m_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_1^{m_0} - 0$, in addition, $C_2^{m*}(t) \rightarrow \partial D$ as $t \rightarrow c_2^m - 0$ for every $m \in \mathbb{N}$;
- 4) there are $k_0, m_0 \in \mathbb{N}$ such that $C_1^{k_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_0^k - 0$ and $C_2^{m_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_2^{m_0} - 0$.

The cases 2) and 3) may be considered similarly, so it is sufficient to consider the cases 1), 2) and 4).

Case 1). Let us assume that, $C_1^{m*}(t) \rightarrow \partial D$ as $t \rightarrow c_1^m - 0$ and $C_2^{m*}(t) \rightarrow \partial D$ as $t \rightarrow c_2^m - 0$ for any $m \in \mathbb{N}$. Then there is $\delta_0 > 0$ such that $d(|C_1^{m*}|) \geq \delta_0 > 0$ and $d(|C_2^{m*}|) \geq \delta_0 > 0$ for any $m \in \mathbb{N}$. Put $P > 0$. Let $U := B(x_0, \delta_0/2)$, and let V be a neighborhood of x_0 which corresponds to Lemma 1 and Remark 2. Since by the assumption $x_m, x'_m \in D \setminus \{x_0\}$, $m = 1, 2, \dots$, we may find a number $m_0 \in \mathbb{N}$ such that $x_m, x'_m \in V$ for any $m \geq m_0$. Observe that, for $m \geq m_0$

$$|C_1^{m*}| \cap \partial V \neq \emptyset, \quad |C_2^{m*}| \cap \partial V \neq \emptyset. \quad (8)$$

Indeed, $x_m \in |C_1^{m*}|$, $x'_m \in |C_2^{m*}|$, and therefore $|C_1^{m*}| \cap V \neq \emptyset \neq |C_2^{m*}| \cap V$ for $m \geq m_0$. Besides that, $\text{diam } V \leq \text{diam } U = r_0$ and, since $d(|C_1^{m*}|) \geq \delta_0 > 0$ and $d(|C_2^{m*}|) \geq \delta_0 > 0$ for any $m \in \mathbb{N}$. Now, by Proposition 2 we obtain the relations (8). Similarly, we may to prove that

$$|C_1^{m*}| \cap \partial U \neq \emptyset, \quad |C_2^{m*}| \cap \partial U \neq \emptyset.$$

Then, by Lemma 1 and Remark 2

$$M(\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})) > P, \quad m \geq m_0. \quad (9)$$

If $p > n$, since D is bounded, for any $\rho \in \text{adm } \Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})$ by the Hölder inequality

$$M(\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})) \leq \int_D \rho^n(x) dm(x) \leq \left(\int_D \rho^p(x) dm(x) \right)^{\frac{n}{p}} \cdot (m(D))^{\frac{p-n}{p}}. \quad (10)$$

Passing in (10) to inf over all $\rho \in \text{adm } \Gamma$, we obtain that

$$\begin{aligned} M(\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})) &\leq \\ &\leq \int_D \rho^n(x) dm(x) \leq (M_p(\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})))^{\frac{n}{p}} \cdot (m(D))^{\frac{p-n}{p}}. \end{aligned} \quad (11)$$

It follows from (9) and (11) that

$$M_p(\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})) \geq P^{\frac{p}{n}} \cdot (m(D))^{\frac{n-p}{n}}. \quad (12)$$

Let us show that the relations (9) and (12) for $p = n$ and $p > n$, respectively, are impossible (in particular, each of them contradicts with the definition of mapping f in (3)–(4)).

Since $\overline{B(z_1, \varepsilon_0)} \cap \overline{B_*(z_2, \varepsilon_1)} = \emptyset$, we may find $\varepsilon_1^* > \varepsilon_0$ for which we still have $B(z_1, \varepsilon_1^*) \cap B_*(z_2, \varepsilon_1) = \emptyset$. Let $\Gamma_* = \Gamma(|C_1^m|, |C_2^m|, D')$. Note that

$$\Gamma_* > \Gamma(S(z_1, \varepsilon_0), S(z_1, \varepsilon_1^*), A(z_1, \varepsilon_0, \varepsilon_1^*)). \quad (13)$$

Indeed, let $\gamma \in \Gamma_*$, $\gamma: [a, b] \rightarrow \mathbb{R}^n$. Since $\gamma(a) \in |C_1^m| \subset B(z_1, \varepsilon_0)$ and $\gamma(b) \in |C_2^m| \subset \overline{\mathbb{R}^n} \setminus B(z_1, \varepsilon_0)$, by Proposition 2 we may find $t_1 \in (a, b)$ such that $\gamma(t_1) \in S(z_1, \varepsilon_0)$. Without loss of generality, we may assume that $|\gamma(t) - z_1| > \varepsilon_0$ for $t > t_1$. Next, since $\gamma(t_1) \in B(z_1, \varepsilon_1^*)$ and $\gamma(b) \in |C_2^m| \subset \mathbb{R}^n \setminus B(z_1, \varepsilon_1^*)$, by Proposition 2 there is $t_2 \in (t_1, b)$ such that $\gamma(t_2) \in S(z_1, \varepsilon_1^*)$. Without loss of generality, we may assume that $|\gamma(t) - z_1| < \varepsilon_0^*$ when $t_1 < t < t_2$. Therefore, $\gamma|_{[t_1, t_2]}$ is a subpath of γ which belongs to $\Gamma(S(z_1, \varepsilon_0), S(z_1, \varepsilon_1^*), A(z_1, \varepsilon_0, \varepsilon_1^*))$. Therefore, the relation (13) is proved.

Let us establish now that

$$\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\}) > \Gamma_f(z_1, \varepsilon_0, \varepsilon_1^*). \quad (14)$$

Indeed, if the path $\gamma: [a, b] \rightarrow D \setminus \{x_0\}$ belongs to $\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})$, then $f(\gamma)$ belongs to D' , and $f(\gamma(a)) \in |C_1^m|$ and $f(\gamma(b)) \in |C_2^m|$, that is, $f(\gamma) \in \Gamma_*$. Then according to the above proof and due to the ratio (13) the path $f(\gamma)$ has a subpath $f(\gamma)^* := f(\gamma)|_{[t_1, t_2]}$, $a \leq t_1 < t_2 \leq b$, which belongs to the family $\Gamma(S(z_1, \varepsilon_1^*), S(z_1, \varepsilon_0), A(z_1, \varepsilon_0, \varepsilon_1^*))$. Then $\gamma^* := \gamma|_{[t_1, t_2]}$ is a subpath of γ and it belongs to $\Gamma_f(z_1, \varepsilon_0, \varepsilon_1^*)$, as required.

In turn, by (14) we have the following:

$$\begin{aligned} M_p(\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})) &\leq \\ &\leq M_p(\Gamma_f(z_1, \varepsilon_0, \varepsilon_1^*)) \leq \int_A Q(y) \cdot \eta^p(|y - z_1|) dm(y), \end{aligned} \quad (15)$$

where $A = A(z_1, \varepsilon_0, \varepsilon_1^*)$ is defined in (2), and η is arbitrary Lebesgue measurable function that satisfies ratio (4) for $r_1 := \varepsilon_0$ and $r_2 := \varepsilon_1^*$. As above, we use the standard agreements:

$a/\infty = 0$ for $a \neq \infty$, $a/0 = \infty$ for $a > 0$ and $0 \cdot \infty = 0$ (see, e.g., [15, 3.I]). Put $\tilde{Q}(y) = \max\{Q(y), 1\}$,

$$I = \int_{\varepsilon_0}^{\varepsilon_1^*} \frac{dt}{t^{\frac{n-1}{p-1}} \tilde{q}_{z_1}^{1/(p-1)}(t)}, \quad (16)$$

where ω_{n-1} denotes the area of the unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n , and $\tilde{q}_{z_1}(t)$ is defined by the relation

$$\tilde{q}_{z_1}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(z_1, r)} \tilde{Q}(y) d\mathcal{H}^{n-1}(y).$$

By the assumption of Theorem 1, there exists a set $E \subset [\varepsilon_0, \varepsilon_1^*]$ of positive linear measure such that $\tilde{q}_{y_1}(t)$ is finite for all $t \in E$. Therefore, $I \neq 0$ in (16). In this case, the function $\eta_0(t) = \frac{1}{I t^{\frac{n-1}{p-1}} \tilde{q}_{z_1}^{1/(p-1)}(t)}$ satisfies the relation (4) for $r_1 := \varepsilon_0$ and $r_2 := \varepsilon_1^*$. Substituting this function in the right-hand side of the inequalities (15) and applying Fubini theorem, we obtain that

$$M_p(\Gamma(|C_1^{m*}|, |C_2^{m*}|, D \setminus \{x_0\})) \leq \frac{\omega_{n-1}}{I^{p-1}} < \infty. \quad (17)$$

The relation (17) contradicts with (9) for $p = n$ and (12) for $p > n$. The above contradiction completes the consideration of the case 1).

Case 2). $C_1^{m*}(t) \rightarrow \partial D$ as $t \rightarrow c_1^m - 0$, but there is $m_0 \in \mathbb{N}$ such that $C_2^{m_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_2^{m_0} - 0$.

We will show that consideration 2) to a certain extent is reduced to case 1). Indeed, since $C_2^{m_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_2^{m_0} - 0$, we may find a sequence $t_k \rightarrow c_2^{m_0} - 0$ such that $C_2^{m_0*}(t_k) \rightarrow x_0$ as $k \rightarrow \infty$. We put $u_k := C_2^{m_0*}(t_k)$ and $v_k := f(C_2^{m_0*}(t_k))$. Also let

$$D^{k*} := C_2^{m_0*}|_{[0, t_k]}, \quad D^k := C_2^{m_0}|_{[0, t_k]}, \quad k = 1, 2, \dots \quad (18)$$

Now we reason in the same way as in case 1), where C_1^{m*} and C_1^m are constructed in the same way, and the role of C_2^{m*} and C_2^m perform D^{m*} and D^m , respectively. Due to the definition C_1^{m*} and D^{m*} , there exists $\delta_0 > 0$ such that $d(|C_1^{m*}|) \geq \delta_0 > 0$ and $d(|D^{m*}|) \geq \delta_0 > 0$ for any $m = 1, 2, \dots$

Let us fix a number $P > 0$. Let $U := B(x_0, \delta_0/2)$, and let V be a neighborhood of the same point x_0 which corresponds to Lemma 1 and Remark 2. Reasoning in the same way as in case 1), we will have

$$M(\Gamma(|C_1^{m*}|, |D^{m*}|, D \setminus \{x_0\})) > P, \quad m \geq m_0 \quad (19)$$

for $p = n$. If $p > n$, since the domain D is bounded,

$$M_p(\Gamma(|C_1^{m*}|, |D^{m*}|, D \setminus \{x_0\})) \geq P^{\frac{p}{n}} \cdot (m(D))^{\frac{n-p}{n}}. \quad (20)$$

Let us show that the relation (19) and (20) for $p = n$ and $p > n$, respectively, are impossible (in particular, each of them contradicts definition of mapping f in (3)–(4)). Really, reasoning in the same way as in case 1), given that the paths C_1^m and D^{m*} lie in disjoint closed balls $\overline{B(z_1, \varepsilon_0)}$ and $\overline{B_*(z_2, \varepsilon_1)}$, we will have that

$$M_p(\Gamma(|C_1^{m*}|, |D^{m*}|, D \setminus \{x_0\})) \leq \frac{\omega_{n-1}}{I^{p-1}} < \infty, \quad (21)$$

where I is defined in (16). The relation (21) contradicts (19) for $p = n$ and (20) for $p > n$. The resulting contradiction completes the consideration case 2).

Case 3) is considered completely similarly to case 2) (differs from it by the designations of paths and corresponding points).

Finally, consider **case 4)**, which also to some extent reduces to case 1).

Assume that there are $k_0, m_0 \in \mathbb{N}$ such that $C_1^{k_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_1^{k_0} - 0$ and $C_2^{m_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_2^{m_0} - 0$. Let D^{*k} and D^k be defined as in (18) with preserving all the relevant notions necessary for them definition.

Next, since $C_1^{k_0*}(t) \rightarrow \{x_0\}$ as $t \rightarrow c_1^{k_0} - 0$, there exists a sequence $s_k \rightarrow c_1^{k_0} - 0$ such that $C_1^{k_0*}(s_k) \rightarrow \{x_0\}$ as $k \rightarrow \infty$. Let us put $w_k := C_1^{k_0*}(s_k)$ and $\omega_k := f(C_1^{k_0*}(s_k))$. Let also

$$E^{k*} := C_1^{k_0*}|_{[0, s_k]}, \quad E^k := C_1^{k_0}|_{[0, s_k]}, \quad k = 1, 2, \dots$$

Now we reason in the same way as in case 1), where the role of C_1^{m*} and C_1^m perform E^{m*} and E^m , respectively, and the role C_2^{m*} and C_2^m perform D^{m*} and D^m , respectively. Due to the definition of E^{k*} and D^{k*} , there exists $\delta_0 > 0$ such that $d(|E^{k*}|) \geq \delta_0 > 0$ and $d(|D^{k*}|) \geq \delta_0 > 0$ for all $k \in \mathbb{N}$.

Put $P > 0$. Let $U := B(x_0, \delta_0/2)$, and let V be a neighborhood of the same point x_0 which corresponds to Lemma 1 and Remark 2. Reasoning in the same way as in case 1), we will have

$$M(\Gamma(|E^{k*}|, |D^{k*}|, D \setminus \{x_0\})) > P, \quad k \geq K_0 \tag{22}$$

in the case $p = n$. If $p > n$, since D is bounded,

$$M_p(\Gamma(|E^{k*}|, |D^{k*}|, D \setminus \{x_0\})) \geq P^{\frac{p}{n}} \cdot (m(D))^{\frac{n-p}{n}}. \tag{23}$$

Observe that the relations (22) and (23) for $p = n$ and $p > n$, respectively, are impossible (in particular, each of them contradicts the definition of the mapping f in (3)–(4)). Indeed, reasoning in the same way as in case 1), given that the paths E^k and D^k lie in disjoint balls $B(z_1, \varepsilon_0)$ and $B_*(z_2, \varepsilon_1)$, we will have that

$$M_p(\Gamma(|E^{k*}|, |D^{k*}|, D \setminus \{x_0\})) \leq \frac{\omega_{n-1}}{I^{p-1}} < \infty, \tag{24}$$

where I is defined in (16). The relation (24) contradicts (22) for $p = n$ and (23) for $p > n$. The resulting contradiction completes the consideration case 4). The theorem is completely proved. \square

From Theorem 1 we immediately have the following consequence, cf. [8, Theorem 1] and also [13, Theorem 6]).

Corollary 2. *Let $n \geq 2, p \geq n$, let D be a domain in $\mathbb{R}^n, x_0 \in D$, and let $f: D \setminus \{x_0\} \rightarrow D', f(D \setminus \{x_0\}) = D' \subset \mathbb{R}^n$, be an open discrete mapping that satisfies conditions (3)–(4) at least in one finite point $y_0 \in C(x_0, f)$. Assume that, $C(x_0, f) \subset \partial D'$, and that for y_0 mentioned above there is $r_0 > 0, r_0 < d_0 = \sup_{y \in f(D)} |y - y_0|$, such that, for any $0 < r_1 < r_2 < r_0$ there is*

a set $E \subset [r_1, r_2]$ of positive linear Lebesgue measure such that the function Q is integrable on $S(y_0, r)$ for each $r \in E$ relative to the $(n - 1)$ -dimensional Hausdorff measure \mathcal{H}^{n-1} on $S(y_0, r)$.

Then f has a continuous extension $\bar{f}: D \setminus \{x_0\} \rightarrow \overline{\mathbb{R}^n}$, the continuity of which should be understood in the sense of the chordal metric h .

In particular, the statement of Corollary 2 is valid if the condition on Q is replaced by a simpler one: $Q \in L^1(D')$.

Proof. Due to the compactness of the space $\overline{\mathbb{R}^n}$, the set $C(x_0, f)$ is not empty. Therefore, the existence of a continuous extension of the mapping f directly follows from Theorem [1](#) taking into account Proposition [5](#). The openness and discreteness of the extended mapping \bar{f} in D follows from Proposition [4](#). \square

4. Proof of Theorem [2](#). Firstly we prove that f has a continuous extension to x_0 . Assume the contrary, i.e, there are $z_1, z_2 \in C(x_0, f)$, $z_1 \neq z_2$. We may assume that $z_1 \neq \infty$. Besides that, at least one point z_1 or z_2 belongs to D' (see Theorem [1](#)). We also may assume that $z_1 \in D'$. Observe that, $\infty \in C(x_0, f)$. Indeed, consider a sequence of punctured balls $U_k := B(x_0, 1/k) \setminus \{x_0\}$, $k = 1, 2, \dots$. Note that, there is $x_k \in U_k$ with $|f(x_k)| \geq k$, because otherwise $|f(x)| \leq k_0$ for some U_{k_0} . The latter implies that f has a continuous extension to x_0 , that contradicts with Remark [1](#). Now, $|f(x_k)| \geq k$ and, consequently, $f(x_k) \rightarrow \infty$ while $x_k \rightarrow x_0$ as $k \rightarrow \infty$. Thus, $\infty \in C(x_0, f)$, as required.

Let [\(1\)](#) holds, i.e., $C(x_0, f) \neq \overline{\mathbb{R}^n}$. Now, there is $z_3 \in \overline{\mathbb{R}^n} \setminus C(x_0, f)$. Since $\infty \in C(x_0, f)$, immediately obtain that $z_3 \neq \infty$. In this case, we show that f omits some a neighborhood V of z_3 in some (sufficiently small) punctured ball $U_0 := B(x_0, r_0) \setminus \{x_0\}$. Otherwise, given $k \in \mathbb{N}$ there is $y_k \in B(z_3, 1/k)$ and $x_k \in U_k := B(x_0, 1/k) \setminus \{x_0\}$ such that $f(x_k) = y_k$. Now, $z_3 \in C(x_0, f)$, contradiction. Now $f|_{U_0}$ has a continuous extension to x_0 , see Remark [1](#). The latter completes the consideration of item [\(1\)](#).

Let [\(2\)](#) holds: $D' \neq \mathbb{R}^n$. Now, there is $z_3 \notin D' \neq \mathbb{R}^n$. There are two possibilities: [\(2.1\)](#) $D' = \mathbb{R}^n \setminus \{z_3\}$, or [\(2.2\)](#) $D' \neq \mathbb{R}^n \setminus \{z_3\}$.

Let [\(2.1\)](#) holds: $D' = \mathbb{R}^n \setminus \{z_3\}$. Let us show that $z_3 \in C(x_0, f)$. Indeed, consider a sequence of punctured balls $U_k := B(x_0, 1/k) \setminus \{x_0\}$, $k = 1, 2, \dots$. Observe that, $f(U_k) = \mathbb{R}^n \setminus \{z_3\}$, because otherwise there is $z_k \neq z_3$ such that $f(U_k) \subset \mathbb{R}^n \setminus \{z_3, z_k\}$. The latter means that f has a continuous extension to x_0 , see Remark [1](#). The latter contradicts with the assumption made above. Now, $z_3, \infty \in C(x_0, f)$, while z_3 and ∞ are boundary points of D' . The latter contradicts with Theorem [1](#).

Let [\(2.2\)](#) holds: $D' \neq \mathbb{R}^n \setminus \{z_3\}$. Now, there is $z_4 \neq z_3$ such that $f(D \setminus \{x_0\}) \subset \mathbb{R}^n \setminus \{z_3, z_4\}$. The latter means that f has a continuous extension to x_0 , see Remark [1](#). The latter contradicts with the assumption made above.

Finally, we consider the case [\(3\)](#): $N(y, f, D \setminus \{x_0\}) < \infty$ at least for one $y \in \mathbb{R}^n$. If $N(y, f, D \setminus \{x_0\}) = 0$, the desired conclusion follows by the item [\(2\)](#). Otherwise, there is $k \in \mathbb{N}$ and $x_1, x_2, \dots, x_k \in D \setminus \{x_0\}$ such that $f^{-1}(y) = \{x_1, x_2, \dots, x_k\}$. In this case, the punctured ball $B(x_0, l_0) \setminus \{x_0\}$, $l_0 = \min_{1 \leq i \leq k} |x_i|$, does not contain the points in $f^{-1}(y)$. In this case, we may apply the result of the item [\(2\)](#) for a domain $D_1 := B(x_0, l_0) \setminus \{x_0\}$ and $f_1 := f|_{D_1}$. Now, by this item, f has a continuous extension to x_0 , that contradicts the assumption made at the beginning of the proof.

The openness and discreteness of the extended mapping $\bar{f}: D \rightarrow \overline{\mathbb{R}^n}$ follows by Proposition [4](#). \square

Some examples of mappings that are involved in the formulation of Theorems [1](#) and [2](#) are presented in [9](#).

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