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The zeta function of some algebraic variables

Anatoliy Pogorui[1,*], Olexander Sarana[2] and Anatoliy Franovskii[1]

[1] Department of Algebra and Geometry, Zhytomyr Ivan Franko State University, Vel. Berdychivska St., 40, 10008, Zhytomyr, Ukraine 10008
[2] Department of Mathematical Analysis, Zhytomyr Ivan Franko State University, Vel. Berdychivska St., 40, 10008, Zhytomyr, Ukraine 10008
* Correspondence: pogor@zu.edu.ua

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Abstract: In this paper, we study the extension of the Riemann zeta function to finite-dimensional commutative algebras and present examples of such extensions for several hypercomplex systems. We also investigate the zeros of the quaternionic zeta function and prove that its nontrivial zeros are spherical.

Keywords: Riemann zeta function, commutative algebra, quaternions, hypercomplex systems

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1. Introduction

Since the Riemann hypothesis is one of the most difficult problems that still remain unsolved, the study of the zeta function is important and relevant. In this paper we consider the Riemann zeta function with arguments belonging to finite-dimensional commutative algebras. Our goal is to describe how the algebraic structure influences the set of zeros and to illustrate this with several examples. The generalization of the zeta function to the algebra of bicomplex numbers was studied in [1]. Extensions of the Riemann zeta function to quaternions and multicomplex numbers were also investigated (see [2] and the bibliography therein). This paper is devoted to the study of the zeta function on split semisimple commutative algebras with identity. This leads to two essentially different situations for zeta functions on algebras over $\mathbb{R}$ and $\mathbb{C}$. We also study the zeros of the zeta function of a quaternionic variable, and these results can possibly be extended to Clifford algebras [3]. We restrict our attention to finite-dimensional *split semisimple commutative algebras*. Algebras with nilpotent elements are not considered in this paper.

2. The Riemann zeta function in finite-dimensional semisimple commutative algebras

Suppose A is a d-dimensional split semisimple commutative algebra over a field $K = \mathbb{R}$ (or $\mathbb{C}$). Then

$$A \cong K^d.$$

There exists a system of orthogonal idempotents $i_1, \ldots, i_d$ such that:

$$i_k i_\ell = 0 \quad (k \neq \ell), \qquad \sum_{k=1}^{d} i_k = 1.$$

Denote by $I_m = \{a i_m | a \in A\}$ the principal ideal generated by i_m, $m = 1, \ldots, d$. It is easily seen that A can be decomposed in the direct sum (the Pierce decomposition)

$$A = I_1 \oplus I_2 \oplus \ldots \oplus I_d.$$

The following lemmas are well known; for their proofs, see, for example, [4].

Lemma 1. *Idempotents i_1, i_2,..., i_d are linearly independent.*

Lemma 2. *If $b \in I_m$ then there exists $k \in K$ such that $b = ki_m$, i.e., the ideal I_m can be represented as follows $I_m = \{ki_m | k \in K\}$.*

From Lemmas 1 and 2 it follows:

Theorem 1. *For any $a \in A$ we have the following decomposition*

$$a = \sum_{m=1}^{d} k_m i_m, \;\; k_m \in K.$$

It is easily verified that $a^l = \sum\limits_{m=1}^{d} k_m^l i_m$.

For $n \in \mathbb{N}$ and $a \in A$, define

$$n^a := \exp(a \ln n).$$

If $a = \sum\limits_{m=1}^{d} k_m i_m$, we have

$$\begin{aligned} n^a &= \exp(a \ln n) = \sum_{l=0}^{\infty} \frac{a^l (\ln n)^l}{l!} \\ &= \sum_{m=1}^{d} \sum_{l=0}^{\infty} \frac{k_m^l (\ln n)^l}{l!} i_m = \sum_{m=1}^{d} n^{k_m} i_m. \end{aligned}$$

The zeta function of a variable $a \in A$.

For $a = \sum\limits_{m=1}^{d} k_m i_m$ the Dirichlet series

$$\sum_{n=1}^{\infty} n^{-a} = \sum_{n=1}^{\infty} \sum_{m=1}^{d} n^{-k_m} i_m = \sum_{m=1}^{d} \sum_{n=1}^{\infty} n^{-k_m} i_m = \sum_{m=1}^{d} \zeta(k_m) i_m,$$

converges componentwise provided

$$\Re(k_m) > 1$$

for all m. In this domain

$$\zeta_A(a) = \sum_{m=1}^{d} \zeta(k_m) i_m.$$

2.1. Meromorphic continuation

We define the meromorphic continuation by

$$\zeta_A^{\text{mer}}(a) = \sum_{m=1}^{d} \zeta(k_m) i_m, \quad k_m \neq 1.$$

The singular set is

$$\{a \in A : k_m = 1\},$$

for some m.

Proposition 1. *The zero set satisfies*

$$\zeta_A(a) = 0 \iff \zeta(k_m) = 0 \quad \forall\, m.$$

If $K = \mathbb{R}$ then the zeros of $\zeta(a)$ occur precisely when each scalar component $\zeta(k_m)$ vanishes. Hence the trivial zeros are obtained when $k_m = -2l_m$, $l_m \in \mathbb{N}$, for all $m \in \{1, 2, \ldots, d\}$.

In the split semisimple complex case, the algebra-valued analogue of the Riemann hypothesis is componentwise equivalent to the classical Riemann hypothesis as follows:

Non-trivial zeros of the zeta function of variable $a \in A$ are located at $a = \frac{1}{2} + i\Sigma$, where $\Sigma = \sum_{m=1}^{d} \sigma_m i_m$, and $\frac{1}{2} + i\sigma_m$ is zero of the complex zeta function $\zeta(z)$, $m \in \{1, 2, \ldots, d\}$.

Thus, this statement does not define a new independent hypothesis. It is equivalent to the classical Riemann hypothesis applied to each component.

Taking into account the functional equation for the zeta function for $s \in \mathbb{C} \setminus \{0, 1\}$ [5]

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s), \tag{1}$$

interpreted as a meromorphic identity, we have

$$\begin{aligned}
\zeta\left(1 - \sum_{m=1}^{d} s_m i_m\right) &= \zeta\left(\sum_{m=1}^{d}(1-s_m)\, i_m\right) = \sum_{m=1}^{d}\zeta\left((1-s_m)\right) i_m \\
&= \sum_{m=1}^{d} 2^{1-s_m}\pi^{-s_m}\cos\left(\frac{\pi s_m}{2}\right)\Gamma\left(s_m\right)\zeta\left(s_m\right) i_m,
\end{aligned}$$

$s_m \in \mathbb{C} \setminus \{0, 1\}$, $m \in \{1, 2, \ldots, d\}$.

Thus, if $a = \frac{1}{2} + i\sum_{m=1}^{d} \sigma_m i_m$ is a nontrivial zero of the zeta function of variable $a \in A$ then $a' = \frac{1}{2} - i\sum_{m=1}^{d} \sigma_m i_m$ is also its nontrivial zero.

If the Riemann hypothesis is false, then there exists a nontrivial zero of the form

$$a = q + i\sum_{m=1}^{d}\sigma_m i_m, \quad q \neq \frac{1}{2},$$

and, by symmetry, the elements

$$q - i\sum_{m=1}^{d}\sigma_m i_m,\ \ 1 - q + i\sum_{m=1}^{d}\sigma_m i_m,\ \ 1 - q - i\sum_{m=1}^{d}\sigma_m i_m,$$

are also its nontrivial zeros.

3. Examples

3.1. The zeta function of a split-complex variable

A split-complex number (or double number) is a number of the form $z = x + yj$, where $x, y \in \mathbb{R}$, $j^2 = 1$, $j \notin \mathbb{R}$.

For $z = x + yj$ the exponential function is defined as

$$\exp(x + yj) = e^x\left(\cosh(y) + j\sinh(y)\right),$$

where $\cosh(y) = \frac{e^y + e^{-y}}{2}$ and $\sinh(y) = \frac{e^y - e^{-y}}{2}$ are hyperbolic cosine and sine respectively.

It follows immediately that

$$\begin{aligned}
\frac{1}{n^z} &= \exp\left(-z\ln n\right) = \exp\left(-x\ln n - jy\ln n\right) \\
&= \exp\left(-x\ln n\right)\left(\cosh\left(y\ln n\right) - jsinh\left(y\ln n\right)\right) \\
&= \frac{1}{n^x}\left(\cosh\left(y\ln n\right) - jsinh\left(y\ln n\right)\right).
\end{aligned}$$

$$\begin{aligned}\zeta(x+yj) &= \sum_{n=1}^{\infty} \frac{1}{n^x} \left(\cosh\left(y \ln n\right) - jsinh\left(y \ln n\right)\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{n^y + n^{-y}}{2n^x} - j\frac{n^y - n^{-y}}{2n^x} \right) \\ &= \frac{1}{2}\left(\zeta(x+y) + \zeta(x-y)\right) + \frac{j}{2}\left(\zeta(x+y) - \zeta(x-y)\right).\end{aligned}$$

The Dirichlet series converges for

$$x+y>1, \quad x-y>1.$$

For other values, the function is defined via analytic continuation.

Zeros of the zeta function of a split-complex variable

Therefore, we can consider the trivial zeros of the zeta function of a split-complex variable. It is easily verified that the trivial zeros are located at points $x=-n-l$, $y=-n+l$, $n,l \in \mathbb{N}$ since in this case we have

$$\zeta(x+yj) = \frac{1}{2}\left(\zeta(-2n) + \zeta(-2l)\right) + \frac{j}{2}\left(\zeta(-2n) - \zeta(-2l)\right) = 0.$$

Specific values of the zeta function of a split-complex variable

Values of $\zeta(x+yj)$ for $x=n+l$, $y=n-l$, $n,l \in \mathbb{N}$ are calculated by using respective values of the zeta function in even natural numbers obtained by Euler as follows

$$\begin{aligned}\zeta\left(n+l+(n-l)j\right) =& \frac{1}{2}\left(\zeta(2n) + \zeta(2l)\right) + \frac{j}{2}\left(\zeta(2n) - \zeta(2l)\right) \\ =& \frac{1}{2}\left(\zeta(2n) + \zeta(2l)\right) + \frac{j}{2}\left(\zeta(2n) - \zeta(2l)\right) \\ =& (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!} + (-1)^{l+1}\frac{(2\pi)^{2l}B_{2l}}{2(2l)!} - j\left((-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n!)} - (-1)^{l+1}\frac{(2\pi)^{2l}B_{2l}}{2(2l)!}\right),\end{aligned}$$

where B_{2n} are the Bernoulli numbers.

For example, for $x=3$, $y=1$, i.e., $n=2$, $l=1$, we have

$$\begin{aligned}\zeta(3+j) &= \frac{1}{2}\left(\zeta(4) + \zeta(2)\right) + \frac{j}{2}\left(\zeta(4) - \zeta(2)\right) \\ &= \frac{1}{2}\left(\frac{\pi^4}{90} + \frac{\pi^2}{6}\right) + \frac{j}{2}\left(\frac{\pi^4}{90} - \frac{\pi^2}{6}\right).\end{aligned}$$

Since $\zeta(0) = -\frac{1}{2}$ we have

$$\zeta(x+xj) = \frac{1}{2}\left(\zeta(2x) + j\zeta(2x)\right) - \frac{1}{2}i_- = \zeta(2x)i_+ - \frac{1}{2}i_-,$$

$$\zeta(x-xj) = \frac{1}{2}\left(\zeta(2x) - j\zeta(2x)\right) + \frac{1}{2}i_+ = \zeta(2x)i_- + \frac{1}{2}i_+,$$

where $i_\pm = \frac{1\pm j}{2}$ are idempotents. Hence,

$$\zeta(x+xj)\zeta(x-xj) = \frac{\zeta(2x)}{2}\left(i_+ - i_-\right) = -\frac{\zeta(2x)}{2}.$$

For $n \in \mathbb{N}$, the values at the points $-n-nj$ and $-n+nj$ should be interpreted via analytic continuation. Thus, the values $\zeta(n+nj)$ and $\zeta(n-nj)$ are zero divisors, and for $n \in N$

$$\zeta(n+nj)\zeta(n-nj) = -\frac{\zeta(2n)}{2} = (-1)^n \frac{(2\pi)^{2n}B_{2n}}{4(2n)!}.$$

In particular

$$\zeta(1+j)\zeta(1-j) = -\frac{\pi^2}{12}.$$

Taking into account the functional equation for the zeta function [5]

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),$$

from $\zeta(x+yj) = \frac{1}{2}\left(\zeta(x+y)+\zeta(x-y)\right) + \frac{j}{2}\left(\zeta(x+y)-\zeta(x-y)\right)$ follows that

$$\begin{aligned}\zeta(x+yj) =& 2^{x+y-1}\pi^{x+y-1}\sin\left(\frac{\pi(x+y)}{2}\right)\Gamma(1-x-y)\zeta(1-x-y)\\ &+2^{x-y-1}\pi^{x-y-1}\sin\left(\frac{\pi(x-y)}{2}\right)\Gamma(1-x+y)\zeta(1-x+y)\\ &+\frac{j}{2}2^{x+y-1}\pi^{x+y-1}\sin\left(\frac{\pi(x+y)}{2}\right)\Gamma(1-x-y)\zeta(1-x-y)\\ &-\frac{j}{2}2^{x-y-1}\pi^{x-y-1}\sin\left(\frac{\pi(x-y)}{2}\right)\Gamma(1-x+y)\zeta(1-x+y).\end{aligned}$$

3.2. The zeta function of a bicomplex variable

A bicomplex number is $w = x+yi+zj+uk = x+yi+(z+ui)j$, where $x, y, z, u \in \mathbb{R}$, $i^2 = j^2 = -1$, $ij = ji = k$.

It is easily seen that $w = z_1 + z_2 j$, where $z_1 = x+yi, z_2 = z+ui$ are complex numbers. Numbers $i_\pm = \frac{1\pm k}{2}$ are idempotents of Bicomplex algebra. For any bicomplex number w we have decomposition

$$w = w_+ + w_-,$$

where $w_+ = wi_+ = z_+i_+, w_- = wi_- = z_-i_-, z_+, z_-$ are complex numbers:

$$z_+ = z_1 - iz_2, \qquad z_- = z_1 + iz_2.$$

It is easily verified that

$$w^k = z_+^k i_+ + z_-^k i_-.$$

The zeta function of a bicomplex variable is as follows

$$\begin{aligned}\zeta(x+yi+zj+uk) &= \sum_{n=1}^{\infty} n^{-w}\\ &= \sum_{n=1}^{\infty} n^{-z_+}i_+ + \sum_{n=1}^{\infty} n^{-z_-}i_-\\ &=\zeta\left(z_+\right)i_+ + \zeta\left(z_-\right)i_-,\end{aligned}$$

where $\zeta\left(z_+\right)$ and $\zeta\left(z_-\right)$ are zeta function of complex variables z_+ and z_- respectively.

3.3. The zeta function of a bihyperbolic variable

Suppose B is the bihyperbolic algebra, that is $B = \{x+ye+zf+ug\}$, where $x, y, z, u \in \mathbb{R}$ and $e, f, g \notin \mathbb{R}$, $e^2 = f^2 = g^2 = 1, ef = fe = g, gf = fg = e$.

It is evident that B is a four-dimensional commutative algebra that has four non-trivial idempotents: $i_1 = \frac{1+e+f+g}{4}, i_2 = \frac{1-e-f+g}{4}, i_3 = \frac{1+e-f-g}{4}, i_4 = \frac{1-e+f-g}{4}$.

It is easily seen that $i_1 + i_2 + i_3 + i_4 = 1$, and $i_k i_l = 0$ for $k \neq l, k, l = 1, 2, 3, 4$. Consider principal ideals $I_m = \{a i_m | a \in B\}$, $m \in \{1, 2, 3, 4\}$ which are generated by idempotents i_m respectively. For any bihyperbolic number w we have decomposition

$$w = r_1 i_1 + r_2 i_2 + r_3 i_3 + r_4 i_4,$$

where $r_k \in \mathbb{R}$, $k \in \{1, 2, 3, 4\}$ are as follows

$$r_1 = x + y + z + u, \quad r_2 = x - y - z + u,$$

$$r_3 = x + y - z - u, \quad r_4 = x - y + z - u.$$

The Dirichlet series converges for $r_m > 1$, $m = 1, 2, 3, 4$. Thus,

$$\zeta(w) = \sum_{n=1}^{\infty} n^{-w} = \zeta(r_1) i_1 + \zeta(r_2) i_2 + \zeta(r_3) i_3 + \zeta(r_4) i_4.$$

4. The zeta-function of a quaternionic variable

Let $\mathbb{H}$ be the skew field of quaternions, that is

$$\mathbb{H} = \{a_0 + i a_1 + j a_2 + k a_3\},$$

where $a_m \in \mathbb{R}$, $m = 0, 1, 2, 3$ are real and $i^2 = j^2 = k^2 = -1$, $ijk = -1$, $ij = k$, $jk = i$, $ki = j$.

In this section we introduce and study the zeta-function of a quaternionic variable. Since the quaternion algebra is not commutative, the approach considered above is not applicable for this case.

For $n \in N$ let us define n^w, where $w = a_0 + i a_1 + j a_2 + k a_3 \in \mathbb{H}$ as follows

$$n^w = \exp(w \ln n) = \exp(a_0 \ln n + \boldsymbol{v} \ln n) = \exp(a_0 \ln n) \exp(\boldsymbol{v} \ln n),$$

where $\boldsymbol{v} = i a_1 + j a_2 + k a_3$.

In the sequel we denote $a_0 = Re(w)$ and $\boldsymbol{v} = Im(w)$. Consider $|\boldsymbol{v}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. By using the generalization of the Euler formula to quaternions [6], we get for $\boldsymbol{v} \neq \boldsymbol{0}$

$$e^{\boldsymbol{v} \ln n} = \cos(|\boldsymbol{v}| \ln n) + \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \sin(|\boldsymbol{v}| \ln n).$$

Hence,

$$n^w = n^{a_0} \left(\cos(|\boldsymbol{v}| \ln n) + \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \sin(|\boldsymbol{v}| \ln n) \right).$$

We first define the function for $\Re(w) > 1$ and extend it by analytic continuation. This definition is consistent with the classical zeta function on each complex slice. We define the quaternionic zeta function via slice extension:

$$\zeta_H(a + v) = Re(\zeta(a + i|v|)) + \frac{v}{|v|} Im(\zeta(a + i|v|)).$$

This construction is related to the theory of slice regular functions; see, for example, [7], [8]. Consider the sphere of radius $r > 0$ of purely imaginary quaternions $S(r) = \{\boldsymbol{v} \in \mathbb{H} : \boldsymbol{v}^2 = -r^2, Re(\boldsymbol{v}) = 0\}$.

Definition 1. All quaternions $w = a + \boldsymbol{v}$, where $a \in \mathbb{R}$ is a constant and $\boldsymbol{v} \in S(r)$ are said to be mutually conjugate via inner automorphisms.

We denote by

$$S^3 = \{q \in \mathbb{H} : |q| = 1\},$$

the group of unit quaternions. A quaternion $w = a + \boldsymbol{v}$, $|\boldsymbol{v}| = r$ lies on a 2-sphere in $\mathbb{H}$ of radius r centered at a_0 and for each conjugate via inner automorphism quaternion $w' = a + \boldsymbol{v}'$ there always exists a quaternion

$q \in S^3$ such that $w' = q^{-1}wq$. Geometrically, this corresponds to a rotation in $\mathbb{R}^3$ sending the direction of $\boldsymbol{v}$ to that of $\boldsymbol{v}'$ by $\boldsymbol{v}' = q^{-1}\boldsymbol{v}q$.

Theorem 2. *Let $s_0 = \frac{1}{2} + i\sigma$, $\sigma \in \mathbb{R} \setminus \{0\}$, be a nontrivial zero of the classical Riemann zeta function. Then the quaternionic zeta function ζ_H vanishes on the whole 2-sphere*

$$[s_0] = \frac{1}{2} + S(|\sigma|) = \left\{ \frac{1}{2} + v : \ v \in Im(\mathbb{H}), \ |v| = |\sigma| \right\}.$$

Equivalently,

$$\zeta_H\left(\frac{1}{2} + \sigma u\right) = 0$$

for every imaginary unit $u \in \mathbb{H}$, $u^2 = -1$.

Proof. By definition of the quaternionic zeta function via slice continuation, for every quaternion $w = a + v$, where $a \in \mathbb{R}$ and $v \in Im(\mathbb{H})$, $v \neq 0$, we have

$$\zeta_H(a+v) = Re(\zeta(a + i|v|)) + \frac{v}{|v|} Im(\zeta(a + i|v|)).$$

Let $v \in Im(\mathbb{H})$ satisfy $|v| = |\sigma|$. Then

$$\zeta_H\left(\frac{1}{2} + v\right) = Re(\zeta\left(\frac{1}{2} + i|\sigma|\right)) + \frac{v}{|\sigma|} Im(\zeta\left(\frac{1}{2} + i|\sigma|\right)).$$

Since $s_0 = \frac{1}{2} + i\sigma$ is a zero of the classical zeta function and the zeros occur in conjugate pairs, we also have

$$\zeta\left(\frac{1}{2} + i|\sigma|\right) = 0.$$

Therefore both its real and imaginary parts vanish:

$$Re(\zeta\left(\frac{1}{2} + i|\sigma|)\right) = 0, \qquad Im(\zeta\left(\frac{1}{2} + i|\sigma|)\right) = 0.$$

Consequently,

$$\zeta_H\left(\frac{1}{2} + v\right) = 0.$$

Thus, ζ_H vanishes at every point of the sphere

$$\frac{1}{2} + S(|\sigma|).$$

This proves the theorem. □

Proposition 2. *For $a \in \mathbb{R}$ and $v \in Im(\mathbb{H})$, $v \neq 0$, one has*

$$\zeta_H(a+v) = 0 \quad \Longleftrightarrow \quad \zeta(a + i|v|) = 0.$$

Proof. Using the slice definition,

$$\zeta_H(a+v) = Re(\zeta(a + i|v|)) + \frac{v}{|v|} Im(\zeta(a + i|v|)).$$

The elements 1 and $v/|v|$ are linearly independent over $\mathbb{R}$. Hence,

$$\zeta_H(a+v) = 0,$$

if and only if $Re(\zeta(a+i|v|)) = 0$ and $Im(\zeta(a+i|v|)) = 0$. This is equivalent to

$$\zeta(a+i|v|) = 0.$$

$\square$

Remark 1. If the Riemann hypothesis is false then the Riemann zeta function $\zeta(z)$, $z \in \mathbb{C}$ has a nontrivial zero $a+\sigma i$, where $\frac{1}{2} \neq a \in (0,1)$. Considering Eq. (1) $1-a-\sigma i$ is also zero of $\zeta(z)$. Similar to the proof of Theorem 2, it can be shown that in this case the function has two spherical zeros $a+S(|\sigma|)$ and $1-a+S(|\sigma|)$.

5. Conclusion

As we can see, the Riemann zeta functions on split semisimple commutative algebras over the field of complex numbers have some similar properties to the complex Riemann zeta function. The extension of the Riemann zeta function to commutative algebras with identity provides a natural generalization of previously studied cases, including bicomplex, hyperbolic, and other hypercomplex systems. The sphericity of the zeros of the quaternion zeta function that we have established is of particular interest for further study of zeta functions of a non-commutative variable. This broader framework enables a systematic exploration of zeta function properties in richer algebraic settings and deepens our understanding of its analytical and structural behavior. As such, it opens new perspectives on the role of the zeta function within number theory and related fields.

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