

Elements of probability theory and measures with values in hypercomplex algebras

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In recent years, the expansion of probability theory and measure theory from real values to values in hypercomplex numbers are actively studied because of their possible applications in mathematics and physics [1] – [5]. In this paper, we extend the notion of probability measure to the case where the measure takes values in the algebra of bihyperbolic numbers [6]. In addition, the concept of the real-valued measure is generalized to the quaternionic-valued measure [7].

The bihyperbolic numbers forms a 4-dimensional algebra over the field of real numbers $\mathbb{W}_4 = \{a_0 + a_1e + a_2f + a_3g, a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$ with basis $\{1, e, f, g\}$ and the following multiplications $e^2 = f^2 = g^2 = 1, ef = fe = g, eg = ge = f, fg = gf = e$.

Lemma 1. [8] *Any bihyperbolic number α can be represented as $\alpha = r_1i_1 + r_2i_2 + r_3i_3 + r_4i_4$, where i_k are idempotents of algebra $\mathbb{W}_4$, $r_k \in \mathbb{R}$, $k = 1, 2, 3, 4$.*

We define on $\mathbb{W}_4$ the relation of partial order $\preceq$ such as $\alpha \preceq \beta \iff \beta - \alpha \in \mathbb{W}_4^+ = \{x_1i_1 + x_2i_2 + x_3i_3 + x_4i_4 \mid x_k \geq 0, k = 1, 2, 3, 4\}$. If $\alpha \preceq \beta$ but $\alpha \neq \beta$, we denote $\alpha \prec \beta$. Let us denote by A_x , the set of all bihyperbolic numbers which are not $\mathbb{W}_4$ -comparable with $x \in \mathbb{W}_4$.

Definition 2. The $\mathbb{W}_4$ -valued modulus of a bihyperbolic number $\alpha = r_1i_1 + r_2i_2 + r_3i_3 + r_4i_4$ is said to be $|\alpha|_{\mathbb{W}_4} = |r_1i_1 + r_2i_2 + r_3i_3 + r_4i_4|_{\mathbb{W}_4} = |r_1|i_1 + |r_2|i_2 + |r_3|i_3 + |r_4|i_4 \in \mathbb{W}_4^+$, where $|r_1|, |r_2|, |r_3|, |r_4|$ are ordinary modules of real numbers.

Definition 3. Let (Ω, Σ) be a measurable space. The function $P_{\mathbb{W}_4} : \Sigma \rightarrow \mathbb{W}_4$ is called a $\mathbb{W}_4$ -valued probability (or bihyperbolic probability) on the σ -algebra of events Σ , if the following conditions hold: 1) $P_{\mathbb{W}_4}(A) \succcurlyeq 0, \forall A \in \Sigma$; 2) $P_{\mathbb{W}_4}(\Omega) = \zeta$, where $\zeta = 1, i_1, i_2, i_3, i_4$; 3) For any sequence $\{A_n, n \geq 1\} \subset \Sigma$ of pairwise incompatible random events we have $P_{\mathbb{W}_4}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P_{\mathbb{W}_4}(A_n)$.

The triplet $(\Omega, \Sigma, P_{\mathbb{W}_4})$ is called a $\mathbb{W}_4$ -probability space.

Each $\mathbb{W}_4$ -valued probability measure can be written in the form $P_{\mathbb{W}_4}(A) = P_1(A)i_1 + P_2(A)i_2 + P_3(A)i_3 + P_4(A)i_4$, where $P_1(A), P_2(A), P_3(A), P_4(A)$ are probabilities.

The topology induced by the bihyperbolic norm generates the Borel σ -algebra $\mathfrak{B}_{\mathbb{W}_4}$ in $\mathbb{W}_4$.

Definition 4. Let $(\Omega, \Sigma, P_{\mathbb{W}_4})$ be a $\mathbb{W}_4$ -probability space. A function $X_{\mathbb{W}_4}(\omega) : \Omega \rightarrow \mathbb{W}_4$ such as $X_{\mathbb{W}_4}^{-1}(A) \in \Sigma$ for each open set A in $\mathbb{W}_4$ is called a $\mathbb{W}_4$ -valued random variable.

Each $\mathbb{W}_4$ -valued random variable $X_{\mathbb{W}_4}(\omega)$ can be written in the following form $X_{\mathbb{W}_4}(\omega) = X_1(\omega)i_1 + X_2(\omega)i_2 + X_3(\omega)i_3 + X_4(\omega)i_4$, where $X_1(\omega), X_2(\omega), X_3(\omega), X_4(\omega)$ are $\mathbb{R}$ -random variables on Ω .

Theorem 5. *The $\mathbb{W}_4$ -valued function $X_{\mathbb{W}_4}(\omega)$ on a measurable space (Ω, Σ) is a $\mathbb{W}_4$ -valued random variable if and only if $\{\omega \in \Omega \mid X_{\mathbb{W}_4}(\omega) \prec x \text{ or } X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$ for all $x \in \mathbb{W}_4$.*

Theorem 6. Let $X_{\mathbb{W}_4}(\omega)$ be a $\mathbb{W}_4$ -valued random variable on $(\Omega, \Sigma, P_{\mathbb{W}_4})$. For $\forall x \in \mathbb{W}_4$ the following conditions are equivalent: $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \preceq x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \succ x \text{ or } X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \succcurlyeq x \text{ or } X_{\mathbb{W}_4}(\omega) \in A_x\} \in \Sigma$; $\{\omega \in \Omega | X_{\mathbb{W}_4}(\omega) \prec x\} \in \Sigma$.

The algebra of quaternions is a structure of the form $\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k, a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$, where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

Definition 7. Let $\mathfrak{M}$ be a σ -algebra of subsets of a non-empty set X . A quaternionic measure ω on a measurable space $(X, \mathfrak{M})$ is a quaternion-valued function on $\mathfrak{M}$ such that for any collection of sets $\{A_n, n \in \mathbb{N}\} \subset \mathfrak{M}$ that $A_n \cap A_m = \emptyset$ whenever $n \neq m$ we have $\omega(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \omega(A_n)$.

Definition 8. The function of the sets $\text{var}[\omega](A) := \sup \sum_{n=1}^{\infty} |\omega(A_n)|$ is defined on the $\mathfrak{M}$, where the supremum is taken for all partitions of A , we call the complete variation ω .

It is clear that $|\omega(A)| \leq \text{var}[\omega](A)$.

Theorem 9. The total variation $\text{var}[\omega]$ of a quaternionic measure ω on a measurable space $(X, \mathfrak{M})$ is a positive measure on $(X, \mathfrak{M})$.

Theorem 10. If ω is a quaternionic measure on a measurable space $(X, \mathfrak{M})$, then $\text{var}[\omega](X) < \infty$.

Definition 11. Let μ be a positive measure and ω be a quaternionic measure on a measurable space $(X, \mathfrak{M})$. We say that ω is absolutely continuous with respect to μ if $\mu(A) = 0$ implies $\omega(A) = 0$ for $A \in \mathfrak{M}$. We write $\omega \ll \mu$.

Definition 12. Given a quaternionic measure ω on a measurable space $(X, \mathfrak{M})$, assume that there is a set $F \in \mathfrak{M}$ such that $\omega(A) = \omega(A \cap F)$ for every $A \in \mathfrak{M}$, we say that ω is concentrated on F . This is equivalent to say that $\omega(A) = 0$ whenever $A \cap F = \emptyset$.

Let ω_1, ω_2 be quaternionic measures on $\mathfrak{M}$ and suppose there exist a pair of disjoint sets F, G such that ω_1 is concentrated on F and ω_2 is concentrated on G . Then we say that ω_1 and ω_2 are mutually singular, and write $\omega_1 \perp \omega_2$.

Theorem 13. Let λ be a signed real σ -finite measure on a measurable space $(X, \mathfrak{M})$ and let ω be a quaternionic measure on $(X, \mathfrak{M})$. Then there exists a unique pair of quaternionic measures ω_a and ω_s such that $\omega = \omega_a + \omega_s$, $\omega_a \ll \lambda$, $\omega_s \perp \lambda$. The pair ω_a, ω_s is called the Lebesgue decomposition of ω w.r.t. λ , where ω_a is the absolutely continuous part and ω_s is the singular part of the decomposition.

Theorem 14. (Radon-Nikodym). Let μ be a positive σ -finite measure on a measurable space $(X, \mathfrak{M})$, let ω be a quaternionic measure on $(X, \mathfrak{M})$ and let ω_a be absolutely continuous part of the Lebesgue decomposition of ω w.r.t. μ . Then there is a measurable quaternionic function $h(x)$ on X such that for every set $A \in \mathfrak{M}$ $\omega_a(A) = \int_A h(x) d\mu$, where $h(x)$ is uniquely defined up to a μ -null set.

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