On equicontinuity of families of mappings by prime ends of variable domains

Evgeny Sevost'yanov[1,2], **Zarina Kovba**[1]
[1] Zhytomyr Ivan Franko State University, Zhytomyr, Ukraine
[2] Institute of Applied Mathematics and Mechanics of the National Academy of Sciences of Ukraine, Sloviansk, Ukraine
esevostyanov2009@gmail.com, *mazhydova@gmail.com*

Let us recall some definitions. A Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation $\int_\gamma \rho(x)\,|dx| \geqslant 1$ holds for all (locally rectifiable) paths $\gamma\in\Gamma$. In this case, we write: $\rho\in\operatorname{adm}\Gamma$. Let $p\geqslant 1$, then *p-modulus* of Γ is defined by the equality $M_p(\Gamma)=\inf\limits_{\rho\in\operatorname{adm}\Gamma}\int_{\mathbb{R}^n}\rho^p(x)\,dm(x)$. We set $M(\Gamma):=M_n(\Gamma)$. Let $x_0\in\mathbb{R}^n$. $0<r_1<r_2<\infty$. $S(x_0,r)=\{x\in\mathbb{R}^n\colon |x-x_0|=r\}$, $B(x_0,r)=\{x\in\mathbb{R}^n\colon |x-x_0|<r\}$ and $A=A(x_0,r_1,r_2)=\{x\in\mathbb{R}^n\colon r_1<|x-x_0|<r_2\}$. Given sets E, $F\subset\overline{\mathbb{R}^n}$ and a domain $D\subset\mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ a family of all paths $\gamma\colon[a,b]\to\overline{\mathbb{R}^n}$ such that $\gamma(a)\in E,\gamma(b)\in F$ and $\gamma(t)\in D$ for $t\in(a,b)$. Let $S_i=S(x_0,r_i)$, $i=1,2$. If $f\colon D\to\mathbb{R}^n$, $y_0\in f(D)$ and $0<r_1<r_2<d_0=\sup\limits_{y\in f(D)}|y-y_0|$, then by $\Gamma_f(y_0,r_1,r_2)$ we denote the family of all paths γ in D such that $f(\gamma)\in\Gamma(S(y_0,r_1),S(y_0,r_2),A(y_0,r_1,r_2))$. Let $Q\colon\mathbb{R}^n\to[0,\infty]$ be a Lebesgue measurable function. We say that f *satisfies Poletsky inverse inequality* at the point $y_0\in f(D)$, if the relation

$$M(\Gamma_f(y_0,r_1,r_2))\leqslant\int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y)\cdot\eta^n(|y-y_0|)\,dm(y)\tag{1}$$

holds for any Lebesgue measurable function $\eta\colon(r_1,r_2)\to[0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1.\tag{2}$$

Later, in the extended space $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y)=|\pi(x)-\pi(y)|$, where π is a stereographic projection $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$ in $\mathbb{R}^{n+1}$. The definitions related to prime ends may be found in [1] and [2]. The *end* of the domain D is the class of equivalent chains of cuts in D. Let K be the end of D in $\mathbb{R}^n$, then the set $I(K)=\bigcap\limits_{m=1}^{\infty}\overline{d_m}$ is called *the impression of* K. Following [2], we say that the end K is *a prime end*, if K contains a chain of cuts $\{\sigma_m\}$ such that $\lim\limits_{m\to\infty}M(\Gamma(C,\sigma_m,D))=0$ for some continuum C in D. The following notation is used: the set of prime ends corresponding to the domain D, is denoted by E_D, and the completion of the domain D by its prime ends is denoted $\overline{D}_P$. The boundary of a domain D in $\mathbb{R}^n$ is said to be *locally quasiconformal* if every $x_0\in\partial D$ has a neighborhood U that admits a quasiconformal mapping φ onto the unit ball $\mathbb{B}^n\subset\mathbb{R}^n$ such that $\varphi(\partial D\cap U)$ is the intersection of $\mathbb{B}^n$ and a coordinate hyperplane. The sequence of cuts σ_m, $m=1,2,\ldots$, is called *regular,* if $\overline{\sigma_m}\cap\overline{\sigma_{m+1}}=\varnothing$ for $m\in\mathbb{N}$ and, in addition, $d(\sigma_m)\to 0$ as $m\to\infty$. If the end K contains at least one regular chain, then K will be called *regular.* We say that a bounded domain D in $\mathbb{R}^n$ is *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in D is regular. Note that space $\overline{D}_P=D\cup E_D$ is metric, which can be demonstrated as follows. If $g\colon D_0 \to D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y \in \overline{D}_P$ we put:

$$\rho(x,y) := |g^{-1}(x) - g^{-1}(y)|\,, \tag{3}$$

where the element $g^{-1}(x)$, $x \in E_D$, is to be understood as some (single) boundary point of D_0, Let D_m, $m = 1, 2, \ldots$, be a sequence of domains in $\mathbb{R}^n$, containing a fixed point A_0. If there exists a ball $B(A_0, \rho)$, $\rho > 0$, belonging to all D_m. then the *kernel* of the sequence D_m, $m = 1, 2, \ldots$, with respect to A_0 is the largest domain D_0 containing A_0 and such that for each compact set E belonging to D_0 there is $N > 0$ such that E belongs to D_m for all $m \geqslant N$. A largest domain is one which contains any other domain having the same property. A sequence of domains D_m, $m = 1, 2, \ldots$, converges to a kernel D_0 if any subsequence of D_m has D_0 as its kernel. Let D_m, $m = 1, 2, \ldots$, be a sequence of domains which converges to a kernel D_0, Then D_m will be called *regular* with respect to D_0, if $D_m \subset D_0$ for all $m \in \mathbb{N}$ and, for every $P_0 \in E_{D_0}$ there is a sequence σ_k, $k = 1, 2, \ldots$, with the following condition: if d_k is a domain in P_0 then there is $M = M(k)$ such that $d_k \cap D_m$ is a non-empty connected set for every $m \geqslant M(k)$. Given $E_1, E_2 \subset \overline{\mathbb{R}^n}$ we set $h(E_1, E_2) = \inf\limits_{x \in E_1, y \in E_2} h(x, y)$. Given $\delta > 0$, domains $D, D_0 \subset \mathbb{R}^n$, $n \geqslant 2$, a compact set E in D_0 and Lebesgue measurable function $Q\colon \mathbb{R}^n \to [0, \infty]$, $Q(y) \equiv 0$ outside D_0, we denote by $\mathfrak{F}^E_{Q,\delta}(D, D_0)$ a class of all open, discrete and closed mappings $f\colon D \to D_0$ of a domain D onto some domain $f(D)$, $E \subset f(D) \subset D_0$, such that (1)–(2) hold for every $y_0 \in \overline{D_0}$ and, in addition, $h(f^{-1}(E), \partial D) \geqslant \delta$.

Theorem. *Let $\delta > 0$, let $D, D_0 \subset \mathbb{R}^n$ be domains, $n \geqslant 2$, let E be a compact set in D_0 and let $Q\colon \mathbb{R}^n \to [0, \infty]$, $Q(y) \equiv 0$ outside D_0, be a Lebesgue measurable function. Assume that,* 1) *no connected component of the boundary of the domain D degenerates into a point,* 2) *D has a weakly flat boundary and* 3) *D_0 is a regular domain. Let $f_m \in \mathfrak{F}^E_{Q,\delta}(D, D_0)$, $m = 1, 2, \ldots$, be a sequence such that:* 4) *every f_m, $m = 1, 2, \ldots$, has a continuous boundary extension $f_m\colon \overline{D} \to \overline{D_0}_P$,* 5) *the sequence of domains $f_m(D)$ is regular with respect to D_0. Assume that, for each point $y_0 \in \overline{D_0}$ and for every $0 < r_1 < r_2 < r_0 := \sup\limits_{y \in D_0} |y - y_0|$ there is a set $E_1 \subset [r_1, r_2]$ of a positive linear Lebesgue measure such that the function Q is integrable with respect to $\mathcal{H}^{n-1}$ over the spheres $S(y_0, r)$ for every $r \in E_1$. Then the family f_m, $m = 1, 2, \ldots$, is uniformly equicontinuous by the metric ρ in $\overline{D_0}_P$ defined by* (3)*. In other words, for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that $\rho(f_m(x), f_m(y)) < \varepsilon$ whenever $|x - y| < \delta$ and for every $m \in \mathbb{N}$. Moreover, there is a subsequence f_{m_k}, $k = 1, 2, \ldots,$ which converges to f uniformly by the metric ρ. In this case, f has a continuous boundary extension $f\colon \overline{D} \to \overline{D_0}_P$ and, besides that, for any $x_0 \in \partial D$ there is $P_0 := f(x_0) \in E_{D_0} = \overline{D_0}_P \setminus D_0$ such that, for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ and $M = M(\varepsilon) \in \mathbb{N}$ such that $\rho(f_{m_k}(x), P_0) < \varepsilon$ for all $x \in B(x_0, \delta) \cap D$ and $k \geqslant M_0$.*

The report was supported by the National Research Foundation of Ukraine, Project number 2025.07/0014, Project name: "Modern problems of Mathematical Analysis and Geometric Function Theory".

1. Kovtonyuk D., Ryazanov V. On the theory of prime ends for space mappings. Ukrainian Math. J., 2015, **67**, No. 4, 528–541.
2. Näkki R. Prime ends and quasiconformal mappings. J. Anal. Math., 1979, **35**, 13–40.