On Koebe's theorem in metric spaces

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Everywhere further (X, d, μ) and (X', d', μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. We will assume that μ is a Borel measure such that $0 < \mu(B) < \infty$ for all balls B in X. Let $y_0 \in X'$, $0 < r_1 < r_2 < \infty$ and $A(y_0, r_1, r_2) = \{y \in X' \colon r_1 < d'(y, y_0) < r_2\}$, $B(y_0, r) = \{y \in X' \colon d'(y, y_0) < r\}, S(y_0, r) = \{y \in X' \colon d'(y, y_0) = r\}$. Given sets E, $F \subset X'$ and a domain $D \subset X'$ denote by $\Gamma(E, F, D)$ the family of all paths $\gamma \colon [a, b] \to X'$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. Given a domain $D \subset X$, a *mapping* $f \colon D \to X'$ is an arbitrary continuous transformation $x \mapsto f(x)$. Let $f \colon D \to X'$. let $y_0 \in f(D)$ and let $0 < r_1 < r_2 < d_0 = \sup\limits_{y \in f(D)} d'(y, y_0)$. Now, we denote by $\Gamma_f(y_0, r_1, r_2)$ the family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$.

Given a path $\gamma \colon [a, b] \to X$ in (X, d), we define its length as the supremum of the sums $\sum\limits_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1}))$ over all partitions $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k = b$ of the segment $[a, b]$. A path γ is called *rectifiable* if its length is finite. Given a family of paths Γ in X, a Borel function $\rho \colon X \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if $\int\limits_{\gamma} \rho\, ds \geqslant 1$ for all (locally rectifiable) $\gamma \in \Gamma$. Given $p \geqslant 1$, the p-modulus of Γ defined as follows: $M_p(\Gamma) = \inf\limits_{\rho \in \operatorname{adm}\Gamma} \int\limits_{X} \rho^p(x)\, d\mu(x)$. Should $\operatorname{adm}\Gamma$ be empty, we set $M_p(\Gamma) = \infty$. Let $Q \colon X' \to [0, \infty]$ be a measurable function. Assume that D and X' have finite Hausdorff dimensions α and $\alpha' \geqslant 1$, respectively. We will say that *f satisfies the inverse Poletsky inequality* at a point $y_0 \in f(D)$, if there is $r_0 = r_0(y_0) > 0$ such that the relation

$$M_\alpha(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{f(D) \cap A(y_0, r_1, r_2)} Q(y) \cdot \eta^{\alpha'}(d'(y, y_0))\, d\mu'(y) \qquad (1)$$

holds for any $0 < r_1 < r_2 < r_0$ any Lebesgue measurable function $\eta \colon (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,. \qquad (2)$$

Let (X, μ) be a metric space with measure μ and of Hausdorff dimension n. For each real number $n \geqslant 1$, we define *the Loewner function* $\Phi_n \colon (0, \infty) \to [0, \infty)$ on X as $\Phi_n(t) = \inf\{M_n(\Gamma(E, F, X)) \colon \Delta(E, F) \leqslant t\}$, where the infimum is taken over all disjoint nondegenerate continua E and F in X and $\Delta(E, F) := \frac{\operatorname{dist}(E,F)}{\min\{d(E), d(F)\}}$. A pathwise connected metric measure space (X, μ) is said to be a *Loewner space* of exponent n, or an n-Loewner space, if the Loewner function $\Phi_n(t)$ is positive for all $t > 0$. Observe that, $\mathbb{R}^n$ and $\mathbb{B}^n \subset \mathbb{R}^n$ are Loewner spaces (see Theorem 8.2 and Example 8.24(a) in [1]). We agree to say that a metric space X is *locally connected* at a point x_0 if for every neighborhood U of x_0 there exists a neighborhood V

of the same point such that the set $V \cap X$ is connected. We say that a space (X, d, μ) is *upper α-regular at a point $x_0 \in X$* if there is a constant $C > 0$ such that

$$\mu(B(x_0, r)) \leqslant Cr^\alpha$$

for the balls $B(x_0, r)$ centered at $x_0 \in X$ with all radii $r < r_0$ for some $r_0 > 0$. We also say that a space (X, d, μ) is *upper α-regular* if the above condition holds at every point $x_0 \in X$.

A space X' is *locally path connected at a point $x_0 \in X'$*, if for an arbitrary neighborhood U of x_0 there exists a neighborhood V of x_0 such that V is path connected. A function $\varphi\colon D \to \mathbb{R}$ integrable on $B(x_0, r)$ is said to have *finite mean oscillation* at a point $x_0 \in D$ (we write $\varphi \in FMO(x_0)$) if

$$\limsup_{\varepsilon \to 0} \frac{1}{\mu(B(x_0, \varepsilon))} \int\limits_{B(x_0,\varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, d\mu(x) < \infty,$$

where

$$\overline{\varphi}_\varepsilon = \frac{1}{\mu(B(x_0, \varepsilon))} \int\limits_{B(x_0,\varepsilon)} \varphi(x) \, d\mu(x).$$

We say that a domain D satisfies *the condition* $\boldsymbol{A}$ if any two pairs of (different) points $a \in D'$, $b \in \overline{D'}$ and $c \in D'$, $d \in \overline{D'}$ may be joined by disjoint paths $\alpha\colon [0,1] \to \overline{D'}$ and $\beta\colon [0,1] \to \overline{D'}$ such that $\alpha(t), \beta(t) \in D'$ for any $t \in [0,1)$. Observe that, the condition $\mathbf{A}$ always hold for domains in $\mathbb{R}^n$, locally connected on the boundary, see e.g. Proposition 1 in [2].

Given metric spaces (X, d, μ) and (X', d', μ') having Hausdorff dimensions $\alpha \geqslant 2$ and $\alpha' \geqslant 2$, respectively, domains $D \subset X$ and $D' \subset X'$, a continuum $E \subset D$, $\delta > 0$ and a measurable function $Q\colon X' \to [0, \infty]$ we denote by $\mathfrak{F}^Q_{E,\delta}(D, D')$ the family of all open discrete mappings $f\colon D \to D'$, satisfying relations (1)–(2) at any point $y_0 \in X'$ such that $d'(f(E)) \geqslant \delta$. The following statement holds.

Theorem. *Let D be a domain in X, while D is locally connected and locally compact α-Loewner space of a Hausdorff dimension $\alpha \geqslant 2$, and let D' be a domain in X' satisfying the condition $\boldsymbol{A}$ which has at least two different boundary points, while $\overline{D'}$ is a compactum in X'. Assume that, X' is a locally path connected and locally compact upper α'-regular metric space of a Hausdorff dimension $\alpha' \geqslant 2$, for which the relation $\mu'(B(z_0, 2r)) \leqslant \gamma \cdot \log^{\alpha'-2} \frac{1}{r} \cdot \mu'(B(z_0, r))$ holds for any $z_0 \in X'$, some $r_0 = r_0(z_0) > 0$ and every $r \in (0, r_0)$. Let $\overline{B(x_0, \varepsilon_1)} \subset D$ for some $\varepsilon_1 > 0$. Assume that, $Q \in FMO(\overline{D'})$. Then there is $r_0 > 0$, which does not depend on f, such that*

$$f(B(x_0, \varepsilon_1)) \supset B(f(x_0), r_0)$$

for all $f \in \mathfrak{F}^Q_{E,\delta}(D, D')$, where $B(f(x_0), r_0) = \{w \in X' \colon d'(w, f(x_0)) < r_0\}$.

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