ON BOUNDARY EXTENSION OF UNCLOSED ORLYCZ-SOBOLEV MAPPINGS

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Abstract

This paper is devoted to the study of the boundary behavior of Orlicz-Sobolev classes that may not preserve the boundary under mapping. Under certain conditions, we show that these mappings have a continuous boundary extension of the definition domain.

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1 Introduction

The present paper is devoted to the study of the boundary behavior of mappings with bounded and finite distortion. Various authors have obtained many results on this topic, see, e.g., [Cr], [MRSY$_1$]–[MRSY$_2$], [Na$_1$]–[Na$_2$] and [Vu]. In particular, relatively recently some authors have studied the Orlicz-Sobolev classes and their connection with the distortion of moduli of families of paths and surfaces, see e.g. [RS$_2$], [KR], [RSS] and [SSP]. We also obtained some results on the boundary behavior of these mappings, see [Sev$_1$] and [Sev$_2$]. In [Sev$_2$], we have established a continuous boundary extension of open discrete Orlicz-Sobolev mappings which are closed (equivalently, boundary preserving). Similar studies have not been carried out for non-closed mappings, and our goal now is to investigate their. Some steps in this direction have already been made in [DS]. Note that, the connection between Orlicz-Sobolev classes and estimates of the distortion of the modulus of families of paths has been established only in some special cases, say, for homeomorphisms, or, more generally, open, discrete and closed mappings (see e.g. [Sev$_2$, Lemma 2, Theorem 5]). The results which we obtain here are new in relation to the results mentioned above.

Recall some definitions. Let U be an open set in $\mathbb{R}^n$. In what follows, $C_0^k(U)$ denotes the space of functions $u : U \to \mathbb{R}$ with a compact support in U, having k partial derivatives with respect to any variable that are continuous in U. Let U be an open set, $U \subset \mathbb{R}^n$, and let $u : U \to \mathbb{R}$ be some function, $u \in L^1_{\rm loc}(U)$. Suppose that, there is a function $v \in L^1_{\rm loc}(U)$ such that

$$\int\limits_U \frac{\partial \varphi}{\partial x_i}(x) u(x)\, dm(x) = -\int\limits_U \varphi(x) v(x)\, dm(x)$$

for any function $\varphi \in C_1^0(U)$. Then we say that the function v is a weak derivative of the first order of the function u with respect to x_i and denote it by $\frac{\partial u}{\partial x_i}(x) := v$.

A function $u \in W_{\rm loc}^{1,1}(U)$ if u has weak derivatives of the first order with respect to each of the variables in U, which are locally integrable in U.

A mapping $f : U \to \mathbb{R}^n$ belongs to the Sobolev class $W_{\rm loc}^{1,1}(U)$, write $f \in W_{\rm loc}^{1,1}(U)$, if all coordinate functions of $f = (f_1, \ldots, f_n)$ have weak partial derivatives of the first order, which are locally integrable in U. We write $f \in W_{\rm loc}^{1,p}(U)$, $p \geqslant 1$, if all coordinate functions of $f = (f_1, \ldots, f_n)$ have weak partial derivatives of the first order, which are locally integrable in U to the degree p.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. The *Orlicz-Sobolev class* $W_{\rm loc}^{1,\varphi}(D)$ is the class of all locally integrable functions f with the first weak derivatives whose gradient ∇f belongs locally in D to the Orlicz class. Note that by the definition $W_{\rm loc}^{1,\varphi} \subset W_{\rm loc}^{1,1}$. For the case when $\varphi(t) = t^p$, $p \geqslant 1$, we write as usual $f \in W_{\rm loc}^{1,p}$.

Later on, we also write $f \in W_{\rm loc}^{1,\varphi}(D)$ for a locally integrable vector-function $f = (f_1, \ldots, f_m)$ of n real variables $x_1, \ldots, x_n$ if $f_i \in W_{\rm loc}^{1,1}(D)$ and

$$\int\limits_G \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty$$

for any domain $G \subset D$ such that $\overline{G} \subset D$, where $|\nabla f(x)| = \sqrt{\sum\limits_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}$.

Recall that a mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of each point $y \in \mathbb{R}^n$ consists of isolated points, and *is open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$.

The boundary of D is called *weakly flat* at the point $x_0 \in \partial D$, if for every $P > 0$ and for any neighborhood U of the point x_0 there is a neighborhood $V \subset U$ of the same point such that $M(\Gamma(E, F, D)) > P$ for any continua $E, F \subset D$ such that $E \cap \partial U \neq \varnothing \neq E \cap \partial V$ and $F \cap \partial U \neq \varnothing \neq F \cap \partial V$. The boundary of D is called weakly flat if the corresponding property holds at any point of the boundary of a domain D.

Given a mapping $f : D \to \mathbb{R}^n$, we denote

$$C(f, x) := \{y \in \overline{\mathbb{R}^n} : \exists\, x_k \in D : x_k \to x, f(x_k) \to y, k \to \infty\} \qquad (1.1)$$

and

$$C(f,\partial D)=\bigcup\limits_{x\in \partial D} C(f, x)\,. \tag{1.2}$$

In what follows, $\mathrm{Int}\,A$ denotes the set of inner points of the set $A\subset \overline{{\Bbb R}^n}$. Recall that, the set $U\subset\overline{{\Bbb R}^n}$ is neighborhood of the point z_0, if $z_0\in \mathrm{Int}\,A$.

We say that a function $\varphi:D\rightarrow{\Bbb R}$ has a *finite mean oscillation* at a point $x_0\in D$, write $\varphi\in FMO(x_0)$, if

$$\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} |\varphi(x)-\overline{\varphi}_{\varepsilon}|\ dm(x)<\infty\,, \tag{1.3}$$

where

$$\overline{\varphi}_{\varepsilon}=\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)} \varphi(x)\,dm(x)\,.$$

Let $Q:{\Bbb R}^n\rightarrow [0,\infty]$ be a Lebesgue measurable function. We set

$$Q^{\,\prime}(x)=\left\{\begin{array}{rr} Q(x), & Q(x)\geqslant 1\,,\\ 1, & Q(x)<1\,. \end{array} \right.$$

Denote by $q^{\,\prime}_{x_0}$ the mean value of $Q^{\,\prime}(x)$ over the sphere $|x-x_0|=r$, that means,

$$q^{\,\prime}_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r}Q^{\,\prime}(x)\,d\mathcal{H}^{n-1}\,. \tag{1.4}$$

Using the inversion $\psi(x)=\frac{x}{|x|^2}$, we may define FMO for $x_0=\infty$.

We say that the boundary ∂D of a domain D in ${\Bbb R}^n$, $n\geqslant 2$, is *strongly accessible at a point $x_0\in\partial D$ with respect to the p-modulus* if for each neighborhood U of x_0 there exist a compact set $E\subset D$, a neighborhood $V\subset U$ of x_0 and $\delta>0$ such that

$$M_p(\Gamma(E,F,D))\geqslant \delta \tag{1.5}$$

for each continuum F in D that intersects ∂U and ∂V. When $p=n$, we will usually drop the prefix in the "p-modulus" when speaking about (1.5).

Assume that, a mapping f has partial derivatives almost everywhere in D. In this case, we set

$$\begin{aligned} l\left(f^{\,\prime}(x)\right)\,&=\,\min\limits_{h\in {\Bbb R}^n \backslash \{0\}} \frac {|f^{\,\prime}(x)h|}{|h|}\,,\\ \Vert f^{\,\prime}(x)\Vert\,&=\,\max\limits_{h\in {\Bbb R}^n \backslash \{0\}} \frac {|f^{\,\prime}(x)h|}{|h|}\,,\\ J(x,f)&=\det f^{\,\prime}(x)\,. \end{aligned} \tag{1.6}$$

and define for any $x\in D$ and $\alpha\geqslant 1$

$$K_{I,\alpha}(x,f)\quad=\quad\left\{ \begin{array}{rr} \frac{|J(x,f)|}{{l\left(f^{\,\prime}(x)\right)}^{\alpha}}, & J(x,f)\ne 0,\\ 1, & f^{\,\prime}(x)=0, \\ \infty, & \text{otherwise} \end{array} \right.\,, \tag{1.7}$$

The following statement holds, cf. [MRSY$_1$, Lemma 5.20, Corollary 5.23], [RS$_1$, Lemma 6.1, Theorem 6.1], [Sm, Lemma 5, Theorem 3] and [DS, Theorem 1.1].

Theorem 1.1. *Let $\alpha\geqslant 1$, let D and D' be bounded domains in $\mathbb{R}^n$, $n\geqslant 3$, let $b\in\partial D$, let $Q:D\rightarrow[0,\infty]$ be a Lebesgue measurable function and let $\varphi:[0,\infty)\rightarrow[0,\infty)$ be an increasing function. Let $f:D\rightarrow D'$ be a bounded open discrete mapping of the class $W_{\rm loc}^{1,\varphi}(D)$, $f(D)=D'$. In addition, assume that $C(f,\partial D)\subset E_*$ for some closed (with respect to D') set $E_*\subset D'$ and $f^{-1}(E_*)=E$ for some closed (with respect to D) set $E\subset D$, besides that,*

1) the set E is nowhere dense in D and D is finitely connected on $E\cup\partial D$, i.e., for any $z_0\in E\cup\partial D$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components,

2) for any neighborhood U of b there is a neighborhood $V\subset U$ of b such that:

2a) $V\cap D$ is connected,

2b) $(V\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$,

3) all components of the set $D'\setminus E_*$ have a strongly accessible boundary with respect to α-modulus,

4) the function φ satisfies the following Calderon condition

$$\int\limits_{1}^{\infty}\left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}}dt<\infty\,. \tag{1.8}$$

Suppose that $K_{I,\alpha}(x,f)\leqslant Q(x)$ a.e. and at least one of the following conditions is satisfied:

5.1) the function Q has a finite mean oscillation at a point b;

5.2) $q_b(r)\ =\ O\left(\left[\log\frac{1}{r}\right]^{n-1}\right)$ as $r\rightarrow 0$;

5.3) the condition

$$\int\limits_{0}^{\delta(b)}\frac{dt}{t^{\frac{n-1}{\alpha-1}}q_b^{\prime\,\frac{1}{\alpha-1}}(t)}=\infty \tag{1.9}$$

holds for some $\delta(b)>0$. Then f has a continuous extension to b.

If the above is true for any point $b\in\partial D$, the mapping f has a continuous extension $\overline{f}:\overline{D}\rightarrow\overline{D'}$, moreover, $\overline{f}(\overline{D})=\overline{D'}$.

For Sobolev classes on the plane, Theorem 1.1 has a somewhat simpler form.

Theorem 1.2. *Let $\alpha\geqslant 1$, let D and D' be bounded domains in $\mathbb{R}^2$, let $b\in\partial D$, let $Q:D\rightarrow[0,\infty]$ be a locally integrable function. Let $f:D\rightarrow D'$ be a bounded open discrete* *mapping of the class* $W_{\rm loc}^{1,1}(D)$, $f(D)=D^{\,\prime}$. *In addition, assume that* $C(f, \partial D)\subset E_*$ *for some closed (with respect to* $D^{\,\prime}$*) set* $E_*\subset D^{\,\prime}$ *and* $f^{\,-1}(E_*)=E$ *for some closed (with respect to* D*) set* $E\subset \overline{D}$, *besides that, the conditions 1)–3) from Theorem 1.1 hold. Suppose also that* $K_{I, \alpha}(x, f)\leqslant Q(x)$ *a.e. and at least one of the following conditions is satisfied: 1) the function* Q *has a finite mean oscillation at a point* b; *2)* $q_b(r)\,=\,O\left(\left[\log{\frac{1}{r}}\right]\right)$ *as* $r\rightarrow 0$; *3) the condition (1.9) holds for some* $\delta(b)>0$. *Then* f *has a continuous extension to* b.

If the above is true for any point $b\in \partial D$, *the mapping* f *has a continuous extension* $\overline{f}:\overline{D}\rightarrow \overline{D^{\,\prime}}$, *moreover,* $\overline{f}(\overline{D})=\overline{D^{\,\prime}}$.

2 Preliminaries

A Borel function $\rho:{\Bbb R}^n\,\rightarrow [0,\infty]$ is called *admissible* for the family Γ of paths γ in ${\Bbb R}^n$, if the relation
$$\int\limits_{\gamma}\rho (x)\,|dx|\geqslant 1$$
holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in {\rm adm} \,\Gamma$. Let $p\geqslant 1$, then *p-modulus* of Γ is defined by the equality
$$M_p(\Gamma)=\inf\limits_{\rho \in \,{\rm adm}\,\Gamma} \int\limits_{{\Bbb R}^n} \rho^p (x)\,dm(x)\,.$$

Let $x_0\in {\Bbb R}^n$, $0<r_1<r_2<\infty$,
$$S(x_0,r) = \{ x\,\in\,{\Bbb R}^n : |x-x_0|=r\}\,, \quad B(x_0, r)=\{x\,\in\,{\Bbb R}^n: |x-x_0|< r\} \tag{2.1}$$
and
$$A=A(x_0, r_1, r_2)=\{ x\,\in\,{\Bbb R}^n : r_1<|x-x_0|<r_2\}\,. \tag{2.2}$$
The most important tool of the paper is the connection between Sobolev (Orlicz-Sobolev) classes and lower (ring) Q-mappings. The theoretical part of this connection is mainly established in [Sev_2].

Let ω be an open set in $\overline{{\Bbb R}^k}:={\Bbb R}^k\cup\{\infty\}$, $k=1,\ldots,n-1$. A (continuous) mapping $S:\omega\rightarrow{\Bbb R}^n$ is called a k-dimensional surface S in ${\Bbb R}^n$. Sometimes we call the image $S(\omega)\subseteq {\Bbb R}^n$ the surface S, too. The number of preimages
$$N(S,y)\ \ =\ \ {\rm card}\ S^{-1}(y)={\rm card}\ \{x\in\omega:\ S(x)=y\} \tag{2.3}$$
is said to be a *multiplicity function* of the surface S at a point $y\in{\Bbb R}^n$. In other words, $N(S,y)$ denotes the multiplicity of covering of the point y by the surface S. It is known that the multiplicity function is lower semi continuous, i.e.,
$$N(S,y)\geqslant\liminf_{m\rightarrow\infty}:N(S,y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \ldots$, such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$, see, e.g., [RR], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure $\mathcal{H}^k$ (see, e.g., [Sa], p. 52).

If $\rho : \mathbb{R}^n \to [0, \infty]$ is a Borel function, then its *integral over a k-dimensional surface* S in $\mathbb{R}^n$, $n \geqslant 2$, is defined by the equality

$$\int\limits_S \rho \, d\mathcal{A} \ := \int\limits_{\mathbb{R}^n} \rho(y) \, N(S, y) \, d\mathcal{H}^k y \, . \tag{2.4}$$

Given a family $\mathcal{S}$ of such k-dimensional surfaces S in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for $\mathcal{S}$, abbr. $\rho \in \operatorname{adm} \mathcal{S}$, if

$$\int\limits_S \rho^k \, d\mathcal{A} \ \geqslant \ 1 \tag{2.5}$$

for every $S \in \mathcal{S}$. Given $p \in [k, \infty)$, the *p-modulus* of $\mathcal{S}$ is the quantity

$$M_p(\mathcal{S}) \ = \inf_{\rho \in \operatorname{adm} \mathcal{S}} \int\limits_{\mathbb{R}^n} \rho^p(x) \, dm(x) \, . \tag{2.6}$$

The next class of mappings is a generalization of quasiconformal mappings in the sense of Gehring's ring definition (see [Ge]; cf. [MRSY$_2$, Chapter 9]). Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 2$. Suppose that $x_0 \in \overline{D} \setminus \{\infty\}$ and $Q\colon D \to (0, \infty)$ is a Lebesgue measurable function. A mapping $f : D \to D'$ is called a *lower Q-mapping at a point x_0 relative to p-modulus* if

$$M_p(f(\Sigma_\varepsilon)) \geqslant \inf_{\rho \in \operatorname{ext}_p \operatorname{adm} \Sigma_\varepsilon} \int\limits_{D \cap A(x_0, \varepsilon, r_0)} \frac{\rho^p(x)}{Q(x)} \, dm(x) \tag{2.7}$$

for every spherical ring $A(x_0, \varepsilon, r_0) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < r_0\}$, $r_0 \in (0, d_0)$, $d_0 = \sup\limits_{x \in D} |x - x_0|$, where Σ_ε is the family of all intersections of the spheres $S(x_0, r)$ with the domain D, $r \in (\varepsilon, r_0)$. If $p = n$, we say that f is a lower Q-mapping at x_0. We say that f is a lower Q-mapping relative to p-modulus in $A \subset \mathbb{R}^n$ if (2.7) is true for all $x_0 \in A$.

Let

$$N(y, f, A) \ = \ \operatorname{card} \{x \in A : f(x) = y\} \ , \tag{2.8}$$

$$N(f, A) \ = \ \sup_{y \in \mathbb{R}^n} N(y, f, A) \, . \tag{2.9}$$

The following lemma holds, see e.g. [Sev$_1$, Lemma 5.1].

Lemma 2.1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $\varphi\colon (0, \infty) \to (0, \infty)$ be a monotone nondecreasing function satisfying (1.8). If $p > n - 1$, then every open discrete mapping $f\colon D \to \mathbb{R}^n$ of class $W^{1,\varphi}_{\rm loc}$ with finite distortion and such that $N(f, D) < \infty$ is a lower Q-mapping relative to the p-modulus at every point $x_0 \in \overline{D}$ for*

$$Q(x) = N(f, D) \cdot K_{I,\alpha}^{\frac{p-n+1}{n-1}}(x, f),$$

$\alpha:=\frac{p}{p-n+1}$, *where the inner dilation* $K_{I,\alpha}(x,f)$ *for* f *at* x *of order* α *is defined by* (1.7), *and the multiplicity* $N(f,D)$ *is defined by the relation (2.9).*

The statement similar to Lemma 2.1 holds for $n=2$, but for Sobolev classes (see e.g. [Sev$_3$, Theorem 4]).

Lemma 2.2. *Let* D *be a domain in* $\mathbb{R}^2$, $n\geqslant 2$, *and let* $p>1$. *Then every open discrete mapping* $f\colon D\rightarrow\mathbb{R}^2$ *of class* $W_{\rm loc}^{1,1}$ *with finite distortion and such that* $N(f,D)<\infty$ *is a lower* Q*-mapping relative to the* p*-modulus at every point* $x_0\in\overline{D}$ *for*

$$Q(x)=N(f,D)\cdot K^{p-1}_{I,\alpha}(x,f),$$

$\alpha:=\frac{p}{p-1}$, *where the inner dilation* $K_{I,\alpha}(x,f)$ *for* f *at* x *of order* α *is defined by* (1.7), *and the multiplicity* $N(f,D)$ *is defined by the relation (2.9).*

Given sets $E,F\subset\overline{\mathbb{R}^n}$ and a domain $D\subset\mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ a family of all paths $\gamma:[a,b]\rightarrow\overline{\mathbb{R}^n}$ such that $\gamma(a)\in E,\gamma(b)\in\ F$ and $\gamma(t)\in D$ for $t\in(a,b)$. Let $S_i=S(x_0,r_i)$, $i=1,2$, where spheres $S(x_0,r_i)$ centered at x_0 of the radius r_i are defined in (2.1). The following statement holds, see [Sev$_2$].

Lemma 2.3. *Let* $x_0\in\partial D$, *let* $f:D\rightarrow\mathbb{R}^n$ *be a bounded, open, discrete, and closed lower* Q*-mapping with respect to* p*-modulus in a domain* $D\subset\mathbb{R}^n$, $Q\in L_{loc}^{\frac{n-1}{p-n+1}}(\mathbb{R}^n)$, $n-1<p$, *and* $\alpha:=\frac{p}{p-n+1}$. *Then, for every* $\varepsilon_0<d_0:=\sup\limits_{x\in D}|x-x_0|$ *and every compact set* $C_2\subset D\setminus B(x_0,\varepsilon_0)$ *there exists* ε_1, $0<\varepsilon_1<\varepsilon_0$, *such that, for each* $\varepsilon\in(0,\varepsilon_1)$ *and each compact* $C_1\subset\overline{B(x_0,\varepsilon)}\cap D$ *the inequality*

$$M_{\alpha}(f(\Gamma(C_1,C_2,D)))\leqslant\int\limits_{A(x_0,\varepsilon,\varepsilon_1)}Q^{\frac{n-1}{p-n+1}}(x)\eta^{\alpha}(|x-x_0|)\,dm(x)\tag{2.10}$$

holds, where $A(x_0,\varepsilon,\varepsilon_1)=\{x\in\mathbb{R}^n:\varepsilon<|x-x_0|<\varepsilon_1\}$ *and* $\eta:(\varepsilon,\varepsilon_1)\rightarrow[0,\infty]$ *is an arbitrary Lebesgue measurable function such that*

$$\int\limits_{\varepsilon}^{\varepsilon_1}\eta(r)\,dr=1\,.\tag{2.11}$$

Let $D\subset\mathbb{R}^n$, let $f:D\rightarrow\mathbb{R}^n$ be a discrete open mapping, $\beta:[a,\,b)\rightarrow\mathbb{R}^n$ be a path, and $x\in\,f^{\,-1}(\beta(a))$. A path $\alpha:[a,\,c)\rightarrow D$ is called a *maximal* f*-lifting* of β starting at x, if (1) $\alpha(a)=x\,$; (2) $f\circ\alpha=\beta|_{[a,\,c)}$; (3) for $c<c'\leqslant b$, there is no a path $\alpha':[a,\,c')\rightarrow D$ such that $\alpha=\alpha'|_{[a,\,c)}$ and $f\circ\alpha'=\beta|_{[a,\,c')}$. Here and below we say that a path $\beta:[a,b)\rightarrow\overline{\mathbb{R}^n}$ converges to the set $C\subset\overline{\mathbb{R}^n}$ as $t\rightarrow b$, if $h(\beta(t),C)=\sup\limits_{x\in C}h(\beta(t),C)\rightarrow 0$ at $t\rightarrow b$. The following is true (see [MRV, Lemma 3.12]).

Proposition 2.1. *Let* $f:D\rightarrow\mathbb{R}^n$, $n\geqslant 2$, *be an open discrete mapping, let* $x_0\in D$, *and let* $\beta:[a,\,b)\rightarrow\mathbb{R}^n$ *be a path such that* $\beta(a)=f(x_0)$ *and such that either* $\lim\limits_{t\rightarrow b}\beta(t)$ *exists,* *or $\beta(t)\to\partial f(D)$ as $t\to b$. Then β has a maximal f-lifting $\alpha:[a,\,c)\to D$ starting at x_0. If $\alpha(t)\to x_1\in D$ as $t\to c$, then $c=b$ and $f(x_1)=\lim\limits_{t\to b}\beta(t)$. Otherwise $\alpha(t)\to\partial D$ as $t\to c$.*

The following statement is proved in [DS, Lemma 2.1].

Lemma 2.4. *Let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $x_0\in\partial D$. Assume that E is closed and nowhere dense in D, and D is finitely connected on E, i.e., for any $z_0\in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components.*

In addition, assume that the following condition holds: for any neighborhood U of x_0 there is a neighborhood $V\subset U$ of x_0 such that:

a) $V\cap D$ is connected,

b) $(V\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$.

Let $x_k,y_k\in D\setminus E$, $k=1,2,\ldots,$ be a sequences converging to x_0 as $k\to\infty$. Then there are subsequences x_{k_l} and y_{k_l}, $l=1,2,\ldots,$ belonging to some sequence of neighborhoods V_l, $l=1,2,\ldots,$ of the point x_0 such that $V_l\subset B(x_0,2^{-l})$, $l=1,2,\ldots,$ and, in addition, any pair x_{k_l} and y_{k_l} may be joined by a path γ_l in $V_l\cap D$, where γ_l contains at most $m-1$ points in E.

The following statement may be found in [Sev_1, Lemmas 1.3 and 1.4].

Proposition 2.2. *Let $Q:\mathbb{R}^n\to[0,\infty]$, $n\geqslant 2$, $n-1<p\leqslant n$, be a Lebesgue measurable function and let $x_0\in\mathbb{R}^n$. Assume that either of the following conditions holds*
(a) $Q\in FMO(x_0)$,
(b) $q_{x_0}(r)\ =\ O\left(\left[\log\frac{1}{r}\right]^{n-1}\right)$ as $r\to 0$,
(c) for some small $\delta_0=\delta_0(x_0)>0$ we have the relations

$$\int\limits_{\delta}^{\delta_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(t)}<\infty,\qquad 0<\delta<\delta_0,$$

and

$$\int\limits_{0}^{\delta_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(t)}=\infty\,.$$

Then there exist a number $\varepsilon_0\in(0,1)$ and a function $\psi:(0,\varepsilon_0)\to[0,\infty)$ such that the relation

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0}Q(x)\cdot\psi^p(|x-x_0|)\;dm(x)=o(I^p(\varepsilon,\varepsilon_0))$$

holds as $\varepsilon\to 0$, where

$$0<I(\varepsilon,\varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt<\infty\qquad\forall\quad\varepsilon\in(0,\varepsilon_1)$$

for some $0<\varepsilon_1<\varepsilon_0$.

3 Main Lemma

Various versions of the lemma given below have been given in publications [IR, Lemma 2.1], [MRSY$_1$, Lemma 5.16], [MRSY$_2$, Theorem 4.6, Theorem 13.1], [RS$_1$, Theorem 5.1], [Sev$_2$] and [Sm, Theorem 1], cf. [Va, Theorem 17.15], [Na$_1$, Section 3], [Vu, Theorem 4.10.II] and [Sr, Theorem 4.2], cf. [DS].

Lemma 3.1. *Assume that, under conditions of Theorem 1.1, instead of 5.1)–5.3) the following condition holds: there is* $\varepsilon_0=\varepsilon_0(b)>0$ *and some positive measurable function* $\psi:(0,\varepsilon_0)\rightarrow (0,\infty)$ *such that*

$$0<I(\varepsilon, \varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt < \infty \tag{3.1}$$

for sufficiently small $\varepsilon\in(0,\varepsilon_0)$ *and, in addition,*

$$\int\limits_{A(b,\varepsilon,\varepsilon_0)} Q(x)\cdot\psi^{\,\alpha}(|x-b|)\ dm(x) = o(I^{\alpha}(\varepsilon,\varepsilon_0))\,, \tag{3.2}$$

where $A:=A(b,\varepsilon,\varepsilon_0)$ *is defined in (2.2). Then* f *has a continuous extension to* b.

Proof. Firstly let us to prove that, under conditions of the lemma, $D\setminus E$ consists of finite number of components $D_1,\ldots, D_s$, $1\leqslant s<\infty$. Indeed, assume the contrary, namely, that $D\setminus E$ consists of infinite number of components $D_1, D_2,\ldots$. Let $z_i\in D_i$, $i=1,2,\ldots$. Since D is bounded, there is a subsequence z_{i_k}, $k_1, 2,\ldots$, which converge to some point $z_0\in \overline{D}$. On the other hand, there is a neighborhood V of the point z_0 such that $(V\cap D)\setminus E$ consists of m components. Thus, there is at least one such a component K intersecting infinitely many components $D_{i_1}, D_{i_2},\ldots$. However, by the assumption there is a neighborhood V of z_0 such that $(V\cap D)\setminus E$ consists of finite number of components $V_1,\ldots, V_m$. Now, there is $m_0\in [1, m]$ such that V_{m_0} intersects infinitely many D_i, contradiction. Thus, $D_1,\ldots, D_s$, $1\leqslant s<\infty$, as required.

Now, let us to prove that f is closed in each D_i, $i=1,2,\ldots$, i.e. $f(S)$ is closed in $f(D_i)$ whenever S is closed in D_i. Since f is open and discrete, it is sufficient to prove that f is boundary preserving, that is $C(f, \partial D_i)\subset \partial f(D_i)$ (see, e.g., [Vu, Theorem 3.3]). Indeed, let $z_k\in D_i$, $k=1,2,\ldots$, let $z_k\rightarrow z_0\in \partial D_i$ and let $f(z_k)\rightarrow w_0$ as $k\rightarrow\infty$. We need to prove that $w_0\in \partial f(D_i)$. Assume the contrary, i.e. $w_0\in f(D_i)$. There are two cases: $z_0\in \partial D$ or $z_0\in D$. 1) In the first case, when $z_0\in \partial D$, we obtain that $w_0\in C(f, \partial D)\subset E_*$. But since $w_0\in f(D_i)$, there is $\zeta_i\in D_i$ such that $f(\zeta_i)=w_0$. Now, since $w_0\in E_*$, we obtain that $\zeta_i\in f^{\,-1}(E_*)=E$ and, consequently, $\zeta_i\not\in D_i$, contradiction. 2) Let us consider the second case, when $z_0\in D$. Now, by the definition of D_i, we have that $z_0\in E$. Now, $w_0\in f(E)\subset E_*$. But since $w_0\in f(D_i)$, there is $\zeta_i\in D_i$ such that $f(\zeta_i)=w_0$. Since $w_0\in E_*$, we have that $\zeta_i\in f^{\,-1}(E_*)=E$ and, consequently, $\zeta_i\not\in D_i$, contradiction.

Let K be a component of $D'\setminus E_*$ consisting $f(D_i)$. Observe that $f(D_i)=K$. Indeed, $f(D_i)\subset K$ by the definition. Let us to prove that $K\subset f(D_i)$. Let us prove this inclusion by the contradiction, i.e., let $a_0\in K\setminus f(D_i)$. Chose $b_0\in f(D_i)$ and join the points b_0 and a_0 by a path $\beta:[0,1]\rightarrow K$. Let $\alpha:[0,c)\rightarrow D$ be a maximal f-lifting of β starting at $c_0:=f^{-1}(b_0)\cap D_i$ (this lifting exists by Proposition 2.1). By the same proposition either one of two cases are possible: $\alpha(t)\rightarrow x_1$ as $t\rightarrow c$, or $\alpha(t)\rightarrow \partial D_i$ as $t\rightarrow c$. In the first case, by Proposition 2.1 we obtain that $c=1$ and $f(\beta(1))=f(x_1)=a_0$ which contradict with the choice of a_0. In the second case, when $\alpha(t)\rightarrow\partial D_i$ as $t\rightarrow c$, we obtain that $f(\beta(c))\in C(f,\partial D_i)\subset E_*$. The latter contradicts the definition of β because β does not contain itself points in E_*.

Observe that, $D'\setminus E_*$ consists of finite number of components. Otherwise, $D'\setminus E_*=\bigcup\limits_{i=1}^{\infty}K_i^{\prime}$, $\varnothing\ne K_i^{\prime}\ne K_j^{\prime}\ne\varnothing$ for $i\ne j$. Let K_i be some component of the set $f^{-1}(K_i^{\prime})$. By the definition $K_i\subset D\setminus E$ and $\varnothing\ne K_i\ne K_j\ne\varnothing$ for $i\ne j$. The latter contradicts the proving above for the number of components of $D\setminus E$.

Now, we proceed directly to the proof of the main statement of the lemma. We apply the approach used under the proof of [DS, Lemma 3.1]. Suppose the opposite, i.e., the conclusion of the lemma is not true. Since D' is bounded, there are at least two sequences x_k, $y_k\in D$, $i=1,2,\ldots$, such that x_k, $y_k\in D$, $k=1,2,\ldots$, $x_k\rightarrow b$, $y_k\rightarrow b$ as $k\rightarrow\infty$, and $f(x_k)\rightarrow y$, $f(y_k)\rightarrow y'$ as $k\rightarrow\infty$, while $y'\ne y$. In particular,

$$|f(x_k)-f(y_k)|\geqslant\delta>0 \tag{3.3}$$

for some $\delta>0$ and all $k\in\mathbb{N}$. By the assumption 2), there exists a sequence of neighborhoods $V_k\subset B(b,2^{-k})$, $k=1,2,\ldots$, such that $V_k\cap D$ is connected and $(V_k\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$.

We note that the points x_k and y_k, $k=1,2,\ldots$, may be chosen such that $x_k,y_k\not\in E$. Indeed, since under condition 1) the set E is nowhere dense in D, there exists a sequence $x_{ki}\in D\setminus E$, $i=1,2,\ldots$, such that $x_{ki}\rightarrow x_k$ as $i\rightarrow\infty$. Put $\varepsilon>0$. Due to the continuity of the mapping f at the point x_k, for the number $k\in\mathbb{N}$ there is a number $i_k\in\mathbb{N}$ such that $|f(x_{ki_k})-f(x_k)|<\frac{1}{2^k}$. So, by the triangle inequality

$$|f(x_{ki_k})-y|\leqslant |f(x_{ki_k})-f(x_k)|+|f(x_k)-y|\leqslant\frac{1}{2^k}+\varepsilon\,,$$

$k\geqslant k_0=k_0(\varepsilon)$, because $f(x_k)\rightarrow y$ as $k\rightarrow\infty$ and by the choice of x_k and y. Therefore, $x_k\in D$ may be replaced by $x_{ki_k}\in D\setminus E$, as required. We may reason similarly for the sequence y_k.

Now, by Lemma 2.4 there are subsequences x_{k_l} and y_{k_l}, $l=1,2,\ldots$, belonging to some sequence of neighborhoods V_l, $l=1,2,\ldots$, of the point b such that $\operatorname{diam}V_l\rightarrow 0$ as $l\rightarrow\infty$ and, in addition, any pair x_{k_l} and y_{k_l} may be joined by a path γ_l in V_l, where γ_l contains at most $m-1$ points in E. Without loss of generality, we may assume that the same sequences x_k and y_k satisfy properties mentioned above. Let $\gamma_k:[0,1]\rightarrow D$, $\gamma_k(0)=x_k$ and $\gamma_k(1)=y_k$, $k=1,2,\ldots$.

Observe that, the path $f(\gamma_k)$ contains not more than $m-1$ points in E_*. In the contrary case, there are at least m such points $b_1=f(\gamma_k(t_1)), b_2=f(\gamma_k(t_2)),\ldots, b_m=f(\gamma_k(t_m))$, $0\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_m\leqslant 1$. But now the points $a_1=\gamma_k(t_1), a_2=\gamma_k(t_2),\ldots, a_m=\gamma_k(t_m)$ are in $f^{-1}(E_*)=E$ and simultaneously belong to γ_k. This contradicts the definition of γ_k.

Let
$$b_1=f(\gamma_k(t_1)), b_2=f(\gamma_k(t_2))\quad,\ldots,\quad b_l=f(\gamma_k(t_1))\,,$$
$$0\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_l\leqslant 1,\qquad 1\leqslant l\leqslant m-1\,,$$
be points in $f(\gamma_k)\cap E_*$. By the relation (3.3) and due to the triangle inequality,
$$\delta\leqslant|f(x_k)-f(y_k)|\leqslant\sum\limits_{k=1}^{l-1}|f(\gamma_k(t_k))-f(\gamma_k(t_{k+1}))|\,. \tag{3.4}$$
It follows from (3.4) that, there is $1\leqslant l=l(k)\leqslant m-1$ such that such that
$$|f(\gamma_k(t_{l(k)}))-f(\gamma_k(t_{l(k)+1})|\geqslant\delta/l\geqslant\delta/(m-1)\,. \tag{3.5}$$
Observe that, the set $G_k:=|f(\gamma_k)|_{(t_{l(k)},t_{l(k)+1})}|$ belongs to $D^{\,\prime}\setminus E_*$, because it does not contain any point in E_*.

Since $D^{\,\prime}\setminus E_*$ consists only of finite components, there exists at least one a component of $D^{\,\prime}\setminus E_*$, containing infinitely many components of G_k. Without loss of generality, going to a subsequence, if need, we may assume that all G_k belong to one component K of $D^{\,\prime}\setminus E_*$.

Due to the compactness of $\overline{{\Bbb R}^n}$, we may assume that the sequence $w_k:=f(\gamma_k(t_{l(k)}))$, $k=1,2,\ldots$, converges to some a point $w_0\in\overline{D^{\,\prime}}$. Let us to show that $w_0\in C(f,\partial D)\subset E_*$. Indeed, there are two cases: either $w_k=f(\gamma_k(t_{l(k)}))\in E_*$ for infinitely many k, or $w_k\not\in E_*$ for infinitely many $k\in{\Bbb N}$. In the first case, the inclusion $w_0\in E_*$ is obvious, because E_* is closed. In the second case, obviously, $w_k=f(x_k)$, but this sequence converges to $y\in C(f,\partial D)\subset E_*$ by the assumption.

By the assumption, each component of the set $D^{\,\prime}\setminus E_*$ has a strongly accessible boundary with respect to α-modulus. In this case, for any neighborhood U of the point $w_0\in\partial K$ there is a compact set $C_0^{\prime}\subset D^{\,\prime}$, a neighborhood V of a point w_0, $V\subset U$, and a number $P>0$ such that
$$M_{\alpha}(\Gamma(C_0^{\prime}, F, K))\geqslant P>0 \tag{3.6}$$
for any continua F, intersecting ∂U and ∂V. Choose a neighborhood U of w_0 with $d(U)<\delta/2(m-1)$, where δ is from (3.3). Let $C_0^{\prime}$ and V be a compact set and a neighborhood corresponding to w_0.

Observe that, G_k contains some a continuum $\widetilde{G}_k$ in K intersecting ∂U and ∂V for sufficiently large k. Indeed, by the construction of G_k, there is a sequence of points $a_{s,k}:=f(\gamma_k(p_s))\rightarrow w_k:=f(\gamma_k(t_{l(k)}))$ as $s\rightarrow\infty$ and $b_{s,k}=f(\gamma_k(q_s))\rightarrow f(\gamma_k(t_{l(k)+1}))$ as $s\rightarrow\infty$, where $t_{l(k)})<p_s<q_s<\gamma_k(t_{l(k)+1}))$ and $a_{s,k}, b_{s,k}\in K$. By the triangle inequality and by (3.5)

$$\begin{gathered}|a_{s,k}-b_{s,k}|\geqslant|b_{s,k}-w_k|-|w_k-a_{s,k}|\\ \geqslant|f(\gamma_k(t_{l(k)+1}))-w_k|-|f(\gamma_k(t_{l(k)+1}))-b_{s,k}|-|w_k-a_{s,k}|\\ \geqslant\delta/(m-1)-|f(\gamma_k(t_{l(k)+1})-b_{s,k})|-|w_k-a_{s,k}|\,.\end{gathered}\tag{3.7}$$

Since $a_{s,k}:=f(\gamma_k(p_s))\rightarrow w_k:=f(\gamma_k(t_{l(k)}))$ as $s\rightarrow\infty$ and $b_{s,k}=f(\gamma_k(q_s))\rightarrow f(\gamma_k(t_{l(k)+1}))$ as $s\rightarrow\infty$, it follows from the latter inequality that there is $s=s(k)\in{\Bbb N}$ such that

$$|a_{s(k),k}-b_{s(k),k}|\geqslant\delta/2(m-1)\,.\tag{3.8}$$

Since V is open, there is some neighborhood U_k of w_k such $U_k\subset V$. Since $a_{s,k}\rightarrow w_k:=f(\gamma_k(t_{l(k)}))$ as $s\rightarrow\infty$, we may assume that $a_{s(k),k}\in V$. Now, we set

$$\widetilde{G}_k:=f(\gamma_k)|_{[p_{s(k)}, q_{s(k)}]}\,.$$

In other words, $\widetilde{G}_k$ is a part of the path $f(\gamma_k)$ between points $a_{s(k),k}$ and $b_{s(k),k}$. Let us to show that $\widetilde{G}_k$ intersects ∂U and ∂V. Indeed, by the mentioned above, $a_{s(k),k}\subset V$, so that $V\cap\widetilde{G}_k\ne\varnothing$. In particular, $U\cap\widetilde{G}_k\ne\varnothing$, because $V\subset U$. On the other hand, by (3.8) $d(\widetilde{G}_k)\geqslant\delta/2(m-1)$, however, $d(U)<\delta/2(m-1)$ by the choice of U. In particular, $d(V)<\delta/2(m-1)$. It follows from this that

$$(\overline{{\Bbb R}^n}\setminus U)\cap\widetilde{G}_k\ne\varnothing\,,\qquad(\overline{{\Bbb R}^n}\setminus V)\cap\widetilde{G}_k\ne\varnothing\,.$$

Now, by [Ku, Theorem 1.I.5, §46]

$$\partial U\cap\widetilde{G}_k\ne\varnothing\,, \partial V\cap\widetilde{G}_k\ne\varnothing\,,$$

as required. Now, by (3.6),

$$M_{\alpha}(\Gamma(\widetilde{G}_k, C_0^{\,\prime}, K))\geqslant P>0\,,\qquad k=1,2,\ldots\,.\tag{3.9}$$

Let us to show that, the relation (3.9) contradicts the definition of f and conditions (3.1)–(3.2). Indeed, let us denote by Γ_k the family of all half-open paths $\beta_k:[a, b)\rightarrow\overline{{\Bbb R}^n}$ such that $\beta_k(a)\in|f(\gamma_k)|$, $\beta_k(t)\in K$ for all $t\in[a, b)$ and, moreover, $\lim\limits_{t\rightarrow b-0}\beta_k(t):=B_k\in C_0^{\,\prime}$. Obviously, by (3.9)

$$M_{\alpha}(\Gamma_k)=M_{\alpha}(\Gamma(\widetilde{G}_k, F, K))\geqslant P>0\,,\qquad k=1,2,\ldots\,.\tag{3.10}$$

Since by the proving above $D\setminus E$ consists of finite number of components, we may assume that all paths $\nabla_k:=\gamma_k|_{[p_{s(k)}, q_{s(k)}]}$ belong to (some) one component D_1 of $D\setminus E$. By the proving above, f is closed in D_1 and $f(D_1)=K$. Consider the family $\Gamma_k^{\,\prime}$ of maximal f-liftings $\alpha_k:[a, c)\rightarrow D$ of the family Γ_k starting at $|\nabla_k|$; such a family exists by Proposition 2.1.

Note that, $C(f,\partial D_1)\subset C(f,\partial D)\cup C(f,E)\subset E_*$. Now, observe that, the situation when $\alpha_k\rightarrow\partial D_1$ as $k\rightarrow\infty$ is impossible because f is closed in D_1 and, consequently, f is boundary preserving (see [Vu, Theorem 3.3]). Therefore, by Proposition 2.1 $\alpha_k\rightarrow x_1\in D$ as $t\rightarrow c-0$, and $c=b$ and $f(\alpha_k(b))=f(x_1)$. In other words, the f-lifting α_k is complete, i.e., $\alpha_k:[a,b]\rightarrow D_1$. Besides that, it follows from that $\alpha_k(b)\in f^{\,-1}(C_0^{\prime})\cap D_1$. Since $f(D_1)=K$, we obtain that $f(f^{\,-1}(C_0^{\prime})\cap D_1)=C_0^{\prime}$. In addition, $f^{\,-1}(C_0^{\prime})\cap D_1$ is a compactum in D_1, see e.g. [Vu, Theorem 3.3]. Thus, there is $r_0^{\prime}>0$ such that

$$d(f^{\,-1}(C_0^{\prime})\cap D_1,\partial D)\geqslant r_0>0\,. \tag{3.11}$$

Moreover, it follows from (3.11) that

$$f^{\,-1}(C_0^{\prime})\subset D\setminus B(b,r_0)\,. \tag{3.12}$$

We may consider that $r_0<\varepsilon_0$, where ε_0 is a number from the conditions of the lemma. Applying now Lemmas 2.1 and 2.3, we obtain for $\alpha:=\frac{p}{p-n+1}$ that

$$\begin{gathered}M_{\alpha}(f(\Gamma_k^{\prime}))=M_{\alpha}(\Gamma(\widetilde{G}_k,C_0^{\prime},K))\\ \leqslant (N(f,D_1))^{\frac{n-1}{p-n+1}}\int\limits_{A(b,2^{\,-k},r_0)}Q(x)\eta^{\,\alpha}(|x-b|)\,dm(x)\end{gathered} \tag{3.13}$$

for sufficiently large $k\in\mathbb{N}$ for any nonnegative Lebesgue measurable function η satisfying the relation (2.11) with $\varepsilon=2^{\,-k}$ and $\varepsilon_0=r_0$. It is worth noting that $N(f,D_1)<\infty$ because f is closed in D_1, see e.g. [MS, Lemma 3.3]. Observe that the function

$$\eta(t)=\left\{\begin{array}{rr}\psi(t)/I(2^{\,-k},r_0), & t\in(2^{\,-k},r_0),\\ 0, & t\in\mathbb{R}\setminus(2^{\,-k},r_0)\,,\end{array}\right.$$

where $I(2^{\,-k},r_0)$ is defined in (3.1), satisfies (2.11). Therefore, by (3.13) and (3.1)–(3.2) we obtain that

$$M_{\alpha}(f(\Gamma_k^{\prime}))=M_{\alpha}(\Gamma(\widetilde{G}_k,C_0^{\prime},K))\leqslant (N(f,D_1))^{\frac{n-1}{p-n+1}}\Delta(k)\,, \tag{3.14}$$

where $\Delta(k)\rightarrow 0$ as $k\rightarrow\infty$. The relations (3.14) together with (3.9) contradict each other, which proves the lemma. $\Box$

Proof of Theorem 1.1 immediately follows by Lemma 3.1 and Proposition 2.2, excluding the equality $f(\overline{D})=\overline{D^{\prime}}$. The proof of the latter repeats the last part of the proof of Theorem 3.1 in [SSD]. $\Box$

Proof of Theorem 1.2 is similar to the proof of Theorem 1.1, but uses Lemma 2.3 instead of Lemma 2.1. In all other respects, the proof of this theorem literally repeats the previous proof. $\Box$

A domain $D\subset\mathbb{R}^n$, $n\geqslant 2$, is called a *uniform* domain with respect to p-modulus, $p\geqslant 1$, if, for each $r>0$, there is $\delta>0$ such that $M_p(\Gamma(F,F^*,D))\geqslant\delta$ whenever F and F^*

are continua of D with $h(F)\geqslant r$ and $h(F^*)\geqslant r$. Let I be some set of indices. Domains D_i, $i\in I$, are said to be *equi-uniform* domains with respect to p-modulus, if, for $r>0$, the modulus condition above is satisfied by each D_i with the same number δ. It should be noted that the proposed concept of a uniform domain has, generally speaking, no relation to definition, introduced for the uniform domain in Martio-Sarvas sense [MSa]. Note that, uniform domains with respect to p-modulus have strongly accessible boundaries with respect to p-modulus, see [SevSkv$_1$, Remark 1].

Remark 3.1. The statement of Lemma 3.1 remains true if condition 3) in this lemma is replaced by the conditions:

($3_{\mathbf{a}}$) the family of components of $D'\setminus E_*$ is *equi-uniform* with respect to p-modulus and, besides that,

($3_{\mathbf{b}}$) there is $\delta_*>0$ such that for each component K of $D'\setminus E_*$ there is a nondegenerate continuum $F\subset K$ such that $d(F)\geqslant\delta_*$ and $d(f^{-1}(F),\partial D)\geqslant\delta_*>0$.

The proof of Remark 3.1 may be given similarly to Remark 4.1 in [DS].

4 Some additional information

Later, in the extended space $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y)=|\pi(x)-\pi(y)|$, where π is a stereographic projection $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$ in $\mathbb{R}^{n+1}$, namely,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2}\sqrt{1+{|y|}^2}}\,,\quad x\ne\infty\ne y \tag{4.1}$$

(see [Va, Definition 12.1]). Further, the closure $\overline{A}$ and the boundary ∂A of the set $A\subset\overline{\mathbb{R}^n}$ we understand relative to the chordal metric h in $\overline{\mathbb{R}^n}$.

Following [Ri, Section II.10], a *condenser* is a pair $E=(A,C)$ where $A\subset\mathbb{R}^n$ is open and C is non–empty compact set contained in A. Given $p\geqslant 1$ and a condenser $E=(A,\,C)$, we set

$$\mathrm{cap}_p\,E\quad=\quad\mathrm{cap}_p\,(A,\,C)\quad=\quad\inf_{u\in W_0(E)}\quad\int\limits_A|\nabla u|^p dm(x)$$

where $W_0(E)=W_0(A,\,C)$ is a family of all non-negative functions $u:A\to\mathbb{R}^1$ such that (1) $u\in C_0(A)$, (2) $u(x)\geq 1$ for $x\in C$, and (3) u is ACL. In the above formula $|\nabla u|=\left(\sum\limits_{i=1}^n(\partial_i u)^2\right)^{1/2}$, and $\mathrm{cap}_p\,E$ is called *p-capacity* of the condenser E, see in [Ri, Section II.10].

The following lemma was proved in [Sev$_1$, Lemma 4.4], cf. [SevSkv$_3$, Lemma 4.2].

Lemma 4.1. *Let $x_0 \in D$, let $n \geqslant 2$, let $p > n-1$ and let $Q : D \to [0, \infty]$ be locally integrable function with degree $\alpha := \frac{n-1}{p-n+1}$. If $f : D \to \overline{\mathbb{R}^n}$ is an open discrete lower Q-mapping at x_0 with respect to p-modulus, then f satisfies*

$$\mathrm{cap}_\beta(f(E)) \leqslant \frac{\omega_{n-1}}{I^{*\,\beta-1}}$$

at the point x_0 with $\beta := \frac{p}{p-n+1}$ and $Q^* = Q^{\frac{n-1}{p-n+1}}$.

Here E is the condenser $(B(x_0, r_2), \overline{B(x_0, r_1)})$, $q^*_{x_0}(r)$ denotes the spherical mean of $Q^{\frac{n-1}{p-n+1}}$ over $|x - x_0| = r$ and $I^* = I^*(r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dt}{t^{\frac{n-1}{\beta-1}} q^{*\,\frac{1}{\beta-1}}_{x_0}(t)}$ for $0 < r_1 < r_2 < \mathrm{dist}\,(x_0, \partial D)$.

In what follows we will need the following auxiliary assertion (see, for example, [MRSY$_2$, Lemma 7.4, ch. 7] for $p = n$ and [Sal, Lemma 2.2] for $p \neq n$.

Proposition 4.1. *Let $x_0 \in \mathbb{R}^n$, $Q(x)$ be a Lebesgue measurable function, $Q : \mathbb{R}^n \to [0, \infty]$, $Q \in L^1_{loc}(\mathbb{R}^n)$. We set $A := A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$ and $\eta_0(r) = \frac{1}{Ir^{\frac{n-1}{p-1}} q^{\frac{1}{p-1}}_{x_0}(r)}$, where $I := I = I(x_0, r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q^{\frac{1}{p-1}}_{x_0}(r)}$ and $q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{|x-x_0|=r} Q(x)\, d\mathcal{H}^{n-1}$ is the integral average of the function Q over the sphere $S(x_0, r)$. Then*

$$\frac{\omega_{n-1}}{I^{p-1}} = \int\limits_A Q(x) \cdot \eta_0^p(|x - x_0|)\ dm(x) \leqslant \int\limits_A Q(x) \cdot \eta^p(|x - x_0|)\ dm(x)$$

for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1$. Moreover, (4.1) holds for similar functions η with $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$ (see, e.g., [Sev$_4$, Remark 3.1]).

Combining Lemma 4.1 with Proposition 4.1, we obtain the following statement.

Lemma 4.2. *Let $x_0 \in D$, let $n \geqslant 2$, let $p > n-1$ and let $Q : D \to [0, \infty]$ be locally integrable function with degree $\alpha := \frac{n-1}{p-n+1}$. If $f : D \to \overline{\mathbb{R}^n}$ is an open discrete lower Q-mapping at x_0 with respect to p-modulus, $p > n-1$, then f satisfies*

$$\mathrm{cap}_\beta(f(E)) \leqslant \int\limits_{A(x_0, r_1, r_2)} Q^{\frac{n-1}{p-n+1}}(x) \cdot \eta^\beta(|x - x_0|)\ dm(x)$$

for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$, where $\beta := \frac{p}{p-n+1}$ and $Q^* = Q^{\frac{n-1}{p-n+1}}$.

Here E is the condenser $(B(x_0, r_2), \overline{B(x_0, r_1)})$, $0 < r_1 < r_2 < \mathrm{dist}\,(x_0, \partial D)$.

The following definition is from [Ri, Section 2, Ch. III]. Let F be a compact set in $\mathbb{R}^n$. We say that F is of *p-capacity zero* if $\mathrm{cap}_p\,(A, F) = 0$ for some bounded open set $A \supset F$. An arbitrary set $E \subset \mathbb{R}^n$ is of p-capacity zero if the same is true for every compact subset of E.

In this case we write $\mathrm{cap}_p\,E=0$ ($\mathrm{cap}\,E=0$ if $p=n$), otherwise $\mathrm{cap}_pE>0$. The following statement was proved in [Sev$_3$, Lemma 4] for $p=n$.

Lemma 4.3. *Let $n-1<p\leqslant n$, let D be a domain in $\mathbb{R}^n$, $n\geq 2$, let $E\subset\overline{\mathbb{R}^n}$ be a compact set of positive capacity for $p=n$, and an arbitrary set for $p\neq n$, let $Q:D\rightarrow[0,\infty]$ be a Lebesgue measurable function, and let $\mathfrak{F}^p_{Q,E}(x_0)$ be a family of open discrete mappings $f:D\rightarrow\overline{\mathbb{R}^n}\setminus E$ satisfying condition*

$$\mathrm{cap}_p(f(\mathcal{E}))\leqslant\int\limits_{A(x_0,r_1,r_2)} Q(x)\cdot\eta^p(|x-x_0|)\;dm(x)$$

for any Lebesgue measurable function $\eta:(r_1,r_2)\rightarrow[0,\infty]$ such that $\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1$, $\mathcal{E}=(B(x_0,r_2),\overline{B(x_0,r_1)})$. Assume that, the relations (3.1)–(3.2) holds with $b=x_0$. Then the family of mappings $\mathfrak{F}^p_{Q,E}(x_0)$ is equicontinuous at the point x_0.

Proof. As we have already noted, this statement was proved earlier for $p=n$, see [Sev$_3$, Lemma 4]. The proof for the case $p\neq n$ literally repeats the proof of Lemma 4.3 in [Sev$_1$] and is therefore omitted. $\Box$

5 The equicontinuity in the closure of a domain

Let us formulate some statements about the equicontinuity of the Sobolev and Orlicz-Sobolev classes in the closure of the domain. For some of our other results on this topic, see, for example, [SevSkv$_1$], [SevSkv$_2$] and [DS].

Given $N\in\mathbb{N}$, $\delta>0$, $\alpha\geqslant 1$, closed sets E_*,F in $\overline{\mathbb{R}^n}$, $n\geqslant 2$, a domain $D\subset\mathbb{R}^n$, a closed (with respect to D) set E in D, an increasing function $\varphi:[0,\infty)\rightarrow[0,\infty)$ and a Lebesgue measurable function $Q:D\rightarrow[0,\infty]$ let us denote by $\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ a (some) family of bounded open discrete mappings $f:D\rightarrow\overline{\mathbb{R}^n}\setminus F$ such that $K_{I,\alpha}(x,f)\leqslant Q(x)$ a.e., $N(f,D)\leqslant N$ and, in addition,

1) $C(f,\partial D)\subset E_*$,

2) for each component K of $D^{\,\prime}_f\setminus E_*$, $D^{\,\prime}_f:=f(D)$, there is a continuum $K_f\subset K$ such that $h(K_f)\geqslant\delta$ and $h(f^{\,-1}(K_f),\partial D)\geqslant\delta>0$,

3) $f^{\,-1}(E_*)=E$.

The following statement holds.

Theorem 5.1. *Let $\alpha\in(n-1,n]$, let D be a bounded domain in $\mathbb{R}^n$, $n\geqslant 2$. Assume that:*

1) the set E is nowhere dense in D, and D is finitely connected on $E\cup\partial D$, i.e., for any $z_0\in E\cup\partial D$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components;

2) for any $x_0\in\partial D$ there is $m=m(x_0)\in\mathbb{N}$, $1\leqslant m<\infty$ such that the following is true: for any neighborhood U of x_0 there is a neighborhood $V\subset U$ of x_0 and such that:

2a) $V\cap D$ is connected,

2b) $(V\cap D)\setminus E$ consists at most of m components.

Let for $\alpha=n$ the set F have positive capacity, and for $n-1<\alpha<n$ it is an arbitrary closed set.

Suppose that, for any $x_0\in\overline{D}$ at least one of the following conditions is satisfied: $3_1)$ a function Q has a finite mean oscillation at x_0; $3_2)$ $q_{x_0}(r)=O\left(\left[\log\frac{1}{r}\right]^{n-1}\right)$ as $r\to 0$; $3_3)$ the condition

$$\int\limits_0^{\delta(x_0)}\frac{dt}{t^{\frac{n-1}{\alpha-1}}{q^{\prime}}_{x_0}^{\frac{1}{\alpha-1}}(t)}=\infty$$

holds for some $\delta(x_0)>0$, where $q^{\prime}_{x_0}(t)$ is defined in (1.4).

4) Assume that, the function φ satisfies Calderon condition (1.8).

Let the family of all components of $D^{\prime}_f\setminus E_*$ is equi-uniform over $f\in\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ with respect to α-modulus. Then every $f\in\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ has a continuous extension to ∂D and the family $\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(\overline{D})$, consisting of all extended mappings $\overline{f}:\overline{D}\to\overline{\mathbb{R}^n}$, is equicontinuous in $\overline{D}$. The equicontinuity must be understood in the sense of the chordal metric h defined in (4.1).

The proof of Theorem 1.1 is based on the following lemma.

Lemma 5.1. *The conclusion of Theorem 5.1 remains valid if, under conditions of this theorem, we replace the assumptions $3_1)$–$3_3)$ by the following assumption: Suppose that, for any $x_0\in\overline{D}$ there is $\varepsilon_0=\varepsilon_0(x_0)>0$ and some positive measurable function $\psi:(0,\varepsilon_0)\to(0,\infty)$ such that*

$$0<I(\varepsilon,\varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt<\infty \tag{5.1}$$

for sufficiently small $\varepsilon\in(0,\varepsilon_0)$ and, in addition,

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)}Q(x)\cdot\psi^{\,\alpha}(|x-x_0|)\;dm(x)=o(I^{\,\alpha}(\varepsilon,\varepsilon_0))\,, \tag{5.2}$$

where $A:=A(x_0,\varepsilon,\varepsilon_0)$ is defined in (2.2).

Proof. Let $x_0\in D$. By Lemma 2.1, every $f\in\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ is a lower Q-mapping relative to the p-modulus at every point $x_0\in\overline{D}$ for $Q(x)=N\cdot K^{\frac{p-n+1}{n-1}}_{I,\alpha}(x,f)$, where $\alpha=\frac{p}{p-n+1}$. Now, by Lemma 4.2 f satisfies the relation

$$\mathrm{cap}_{\alpha}(f(\mathcal{E}))\leqslant\int\limits_{A(x_0,r_1,r_2)}N^{\frac{n-1}{p-n+1}}\cdot K_{I,\alpha}(x,f)\eta^{\alpha}(|x-x_0|)\,dm(x)$$
$$\leqslant N^{\frac{n-1}{p-n+1}}\cdot\int\limits_{A(x_0,r_1,r_2)}Q(x)\eta^{\alpha}(|x-x_0|)\,dm(x)$$

for any Lebesgue measurable function $\eta:(r_1,r_2)\rightarrow[0,\infty]$ such that $\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1$. Now, the equicontinuity of the family $\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ at inner points $x_0\in D$ follows by Lemma 4.3.

Since any component of $D^{\,\prime}_f\setminus E_*$ is uniform with respect to α-modulus, it has strongly accessible boundary with respect to α-modulus (see [SevSkv$_1$, Remark 1]). Now the continuous extension of any $f\in\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ to ∂D follows from Lemma 3.1 and Remark 3.1. It remains to prove the equicontinuity of the extended family $\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(\overline{D})$ in ∂D.

Suppose the opposite. Then there is $x_0\in\partial D$, $\varepsilon_0>0$, a sequence $x_k\rightarrow x_0$, $x_k\in\overline{D}$, and $f_k\in\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ such that

$$h(f_k(x_k),f_k(x_0))\geqslant\varepsilon_0\,. \tag{5.3}$$

Since f_k has a continuous extension to ∂D, we may consider that $x_k\in D$. In addition, it follows from (5.3) that we may find a sequence $x^{\,\prime}_k\in D$, $k=1,2,\ldots$, such that $x^{\,\prime}_k\rightarrow x_0$ and

$$h(f_k(x_k),f_k(x^{\,\prime}_k))\geqslant\varepsilon_0/2\,. \tag{5.4}$$

By the assumption 2), there exists a sequence of neighborhoods $V_k\subset B(x_0,2^{-k})$, $k=1,2,\ldots$, such that $V_k\cap D$ is connected and $(V_k\cap D)\setminus E$ consists of m components, $1\leqslant m<\infty$.

Arguing similarly to the proof of Lemma 3.1, we may consider that $x_k,x^{\,\prime}_k\not\in E$. Now, by Lemma 2.4 there are subsequences x_{k_l} and $x^{\,\prime}_{k_l}$, $l=1,2,\ldots$, belonging to some sequence of neighborhoods V_l, $l=1,2,\ldots$, of the point x_0 such that $\mathrm{diam}\,V_l\rightarrow 0$ as $l\rightarrow\infty$ and, in addition, any pair x_{k_l} and $x^{\,\prime}_{k_l}$ may be joined by a path γ_l in $V_l\cap D$, where γ_l contains at most $m-1$ points in E. Without loss of generality, we may assume that the same sequences x_k and y_k satisfy properties mentioned above. Let $\gamma_k:[0,1]\rightarrow D$, $\gamma_k(0)=x_k$ and $\gamma_k(1)=x^{\,\prime}_k$, $k=1,2,\ldots$.

Observe that, the path $f_k(\gamma_k)$ contains not more than $m-1$ points in E_*. Let

$$b_1=f_k(\gamma_k(t_1)),b_2=f_k(\gamma_k(t_2))\quad,\ldots,\quad b_l=f_k(\gamma_k(t_l))\,,$$
$$t_0:=0\leqslant t_1\leqslant t_2\leqslant\ldots\leqslant t_l\leqslant 1=t_{l+1},\qquad 0\leqslant l\leqslant m-1\,,$$

be points in $f_k(\gamma_k)\cap E_*$. By the relation (5.4) and due to the triangle inequality,

$$\varepsilon_0/2\leqslant h(f_k(x_k),f_k(x^{\,\prime}_k))\leqslant\sum\limits_{r=0}^{l}h(f_k(\gamma_k(t_r)),f_k(\gamma_k(t_{r+1}))\,. \tag{5.5}$$

It follows from (5.5) that, there is $1\leqslant r=r(k)\leqslant m-1$ such that

$$h(f(\gamma_k(t_{r(k)})), f_k(\gamma_k(t_{r(k)+1})))\geqslant \varepsilon_0/(2(l+1))\geqslant \varepsilon_0/(2m)\,. \qquad (5.6)$$

Observe that, the set $G_k:=|f(\gamma_k)|_{(t_{r(k)}, t_{r(k)+1})}|$ belongs to $D^{\,\prime}\setminus E_*$ and that G_k contains some a continuum $\widetilde{G}_k$ with $h(\widetilde{G}_k)\geqslant \varepsilon_0/(4m)$ for any $k\in {\Bbb N}$.

Since $\widetilde{G}_k$ is a continuum in $D^{\,\prime}_{f_k}\setminus E_*$, there is a component K_k of $D^{\,\prime}_{f_k}\setminus E_*$, containing K_k. Let us apply the definition of equi-uniformity for he sets $\widetilde{G}_k$ and K_{f_k} in K_k (here K_{f_k} is a continuum from the definition of the class $\frak{R}^{E_*, E, F, N}_{\varphi, Q, \delta, \alpha}(D)$, in particular, $h(K_{f_k})\geqslant \delta$). Due to this definition, for the number $\delta_*:=\min\{\delta, \varepsilon/4m\}>0$ there is $P>0$ such that

$$M_p(\Gamma(\widetilde{G}_k, K_{f_k}, K_k))\geqslant P>0\,,\qquad k=1,2,\ldots\,. \qquad (5.7)$$

Let us to show that, the relation (5.7) contradicts with the definition of the class $\frak{R}^{E_*, E, F, N}_{\varphi, Q, \delta, \alpha}(D)$ together with conditions (3.1)–(3.2). Indeed, let us denote by Γ_k the family of all half-open paths $\beta_k:[a, b)\rightarrow \overline{{\Bbb R}^n}$ such that $\beta_k(a)\in \widetilde{G}_k$, $\beta_k(t)\in K_k$ for all $t\in [a, b)$ and, moreover, $\lim\limits_{t\rightarrow b-0}\beta_k(t):=B_k\in K_{f_k}$. Obviously, by (5.7)

$$M_p(\Gamma_k)=M_p(\Gamma(\widetilde{G}_k, K_{f_k}, K_k))\geqslant P>0\,,\qquad k=1,2,\ldots\,. \qquad (5.8)$$

As under the proof of Lemma 3.1, we may prove that $D\setminus E$ consists of finite number of components $D_1,\ldots, D_s$, $1\leqslant s<\infty$, while f_k is closed in each D_i, $i=1,2,\ldots$, i.e. $f_k(E)$ is closed in $f_k(D_i)$ whenever E is closed in D_i. We also may show that $f_k(D_i)=K$ for some $i\in {\Bbb N}$. Without loss of generality, we may consider that $\nabla_k:=\gamma_k|_{[p_{s(k)}, q_{s(k)}]}$ belong to (some) one component D_1 of $D\setminus E$ and $f_k(D_1)=K$, where $t_{r(k)}<p_{s(k)}<q_{s(k)}<t_{r(k)+1}$, $a_{s,k}:=f_k(\gamma_k(p_s))\rightarrow w_k:=f_k(\gamma_k(t_{r(k)}))$ as $s\rightarrow\infty$ and $b_{s,k}=f_k(\gamma_k(q_s))\rightarrow f_k(\gamma_k(t_{r(k)+1}))$ as $s\rightarrow\infty$. Consider the family $\Gamma^{\,\prime}_k$ of all maximal f_k-liftings $\alpha_k:[a, c)\rightarrow D$ of the family Γ_k starting at $|\nabla_k|$; such a family exists by Proposition 2.1.

Observe that, the situation when $\alpha_k\rightarrow \partial D_1$ as $k\rightarrow\infty$ is impossible. Suppose the opposite: let $\alpha_k(t)\rightarrow \partial D_1$ as $t\rightarrow c$. We choose an arbitrary sequence $\varphi_m\in [0, c)$ such that $\varphi_m\rightarrow c-0$ as $m\rightarrow\infty$. Since the space $\overline{{\Bbb R}^n}$ is compact, the boundary ∂D_1 is also compact as a closed subset of the compact space. Then there exists $w_m\in \partial D_1$ such that

$$h(\alpha_k(\varphi_m), \partial D_1)=h(\alpha_k(\varphi_m), w_m)\rightarrow 0\,,\qquad m\rightarrow \infty\,. \qquad (5.9)$$

Due to the compactness of ∂D_1, we may assume that $w_m\rightarrow w_0\in\partial D_1$ as $m\rightarrow\infty$. Therefore, by the relation (5.9) and by the triangle inequality

$$h(\alpha_k(\varphi_m), w_0)\leqslant h(\alpha_k(\varphi_m), w_m)+h(w_m, w_0)\rightarrow 0\,,\qquad m\rightarrow\infty\,. \qquad (5.10)$$

There are two cases: $w_0\in \partial D$, or $w_0\in D$. In the first case,

$$f_k(\alpha_k(\varphi_m))=\beta_k(\varphi_m)\rightarrow \beta(c)\,,\quad m\rightarrow\infty\,, \qquad (5.11)$$

because by the construction the path $\beta_k(t)$, $t\in[a,b]$, lies in $K_k\subset D'_{f_k}\setminus E_*$ together with its finite ones points. At the same time, by (5.10) and (5.11) we have that $\beta_k(c)\in C(f_k,\partial D)\subset E_*$ by the definition of the class $\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ and because $C(f_k,\partial D)\subset E_*$. The inclusions $\beta_k\subset D'\setminus E_*$ and $\beta_k(c)\in E_*$ contradict each other.

In the second case, when $w_0\in D$, we have that $w_0\in E$. However, $f_k(\alpha_k(\varphi_m))=\beta_k(\varphi_m)\rightarrow \beta_k(c)$ and consequently, $\beta_k(c)\in f_k(E)$. But now $\beta_k(c)\in E_*$ because $f_k^{-1}(E_*)=E$. The latter contradicts the definition of β_k. Thus, the case $\alpha_k\rightarrow\partial D_1$ as $k\rightarrow\infty$ is impossible, as required.

Therefore, by Proposition 2.1 $\alpha_k\rightarrow x_1\in D_1$ as $t\rightarrow c-0$, and c_b and $f_k(\alpha_k(b))=f_k(x_1)$. In other words, the f_k-lifting α_k is complete, i.e., $\alpha_k:[a,b]\rightarrow D$. Besides that, it follows from that $\alpha_k(b)\in f_k^{-1}(K_{f_k})$.

Again, by the definition of the class $\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$,

$$h(f_k^{-1}(K_{f_k}),\partial D)\geqslant\delta>0\,. \tag{5.12}$$

Since $x_0\ne\infty$, it follows from (5.12) that

$$f_k^{-1}(K_{f_k})\subset D\setminus B(x_0,r_0) \tag{5.13}$$

for any $k\in\mathbb{N}$ and some $r_0>0$. Let k be such that $2^{-k}<\varepsilon_0$. We may consider that $r_0<\varepsilon_0$, where ε_0 is a number from the conditions of the lemma. Applying now Lemmas 2.1 and 2.3, we obtain for $\alpha:=\frac{p}{p-n+1}$ that

$$M_{\alpha}(f(\Gamma_k'))=M_{\alpha}(\Gamma(\widetilde{G}_k,C_0',K))\leqslant N^{\frac{n-1}{p-n+1}}\int\limits_{A(x_0,2^{-k},r_0)}Q(x)\eta^{\alpha}(|x-x_0|)\,dm(x) \tag{5.14}$$

for sufficiently large $k\in\mathbb{N}$ and for any nonnegative Lebesgue measurable function η satisfying the relation (2.11) with $\varepsilon=2^{-k}$ and $\varepsilon_0=r_0$. Observe that the function

$$\eta(t)=\begin{cases}\psi(t)/I(2^{-k},r_0), & t\in(2^{-k},r_0),\\ 0, & t\in\mathbb{R}\setminus(2^{-k},r_0)\,,\end{cases}$$

where $I(2^{-k},r_0)$ is defined in (3.1), satisfies (2.11) for $\varepsilon:=2^{-k}$, $\varepsilon_0:=r_0$. Therefore, by (5.14) and (3.1)–(3.2) we obtain that

$$M_{\alpha}(f(\Gamma_k'))=M_{\alpha}(\Gamma(\widetilde{G}_k,C_0',K))\leqslant N^{\frac{n-1}{p-n+1}}\cdot\Delta(k)\,, \tag{5.15}$$

where $\Delta(k)\rightarrow 0$ as $k\rightarrow\infty$. The relations (5.15) together with (5.7) contradict each other, which proves the lemma. $\Box$

Proof of Theorem 5.1 directly follows from Lemmas 5.1 and Proposition 2.2. $\Box$

We also may formulate the corresponding statement for Sobolev space on the plane, cf. Theorem 1.2.

Given $N\in\mathbb{N}$, $\delta>0$, $\alpha\geqslant 1$, closed sets E, E_*, F in $\overline{\mathbb{R}^n}$, $n\geqslant 2$, a domain $D\subset\mathbb{C}$ and a Lebesgue measurable function $Q:D\rightarrow[0,\infty]$ let us denote by $\mathfrak{R}_{Q,\delta,\alpha}^{E_*,E,F,N}(D)$ the family of all bounded open discrete mappings $f:D\rightarrow\overline{\mathbb{C}}\setminus F$ such that $K_{I,\alpha}(x,f)\leqslant Q(x)$ a.e., $N(f,D)\leqslant N$ and, in addition,

1) $C(f,\partial D)\subset E_*$,

2) for each component K of $D_f^{\prime}\setminus E_*$, $D_f^{\prime}:=f(D)$, there is a continuum $K_f\subset K$ such that $h(K_f)\geqslant\delta$ and $h(f^{-1}(K_f),\partial D)\geqslant\delta>0$,

3) $f^{-1}(E_*)=E$.

The following statement holds.

Theorem 5.2. *Let $\alpha\in(1,2]$, let D be a bounded domain in $\mathbb{C}$. Assume that:*

1) the set E is nowhere dense in D, and D is finitely connected on $E\cup\partial D$, i.e., for any $z_0\in E\cup\partial D$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components;

2) for any $x_0\in\partial D$ there is $m=m(x_0)\in\mathbb{N}$, $1\leqslant m<\infty$ such that the following is true: for any neighborhood U of x_0 there is a neighborhood $V\subset U$ of x_0 and such that:

2a) $V\cap D$ is connected,

2b) $(V\cap D)\setminus E$ consists at most of m components.

Let for $\alpha=2$ the set F have positive capacity, and for $1<\alpha<2$ it is an arbitrary closed set.

Suppose that, for any $x_0\in\partial D$ at least one of the following conditions $3_1)$–$3_3)$ of Theorem 5.1 is satisfied.

Let the family of all components of $D_f^{\prime}\setminus E_*$ is equi-uniform over $f\in\mathfrak{R}_{Q,\delta,\alpha}^{E_*,E,F,N}(D)$ with respect to α-modulus. Then every $f\in\mathfrak{R}_{Q,\delta,\alpha}^{E_*,E,F,N}(D)$ has a continuous extension to ∂D and the family $\mathfrak{R}_{Q,\delta,\alpha}^{E_*,E,F,N}(\overline{D})$, consisting of all extended mappings $\overline{f}:\overline{D}\rightarrow\overline{\mathbb{C}}$, is equicontinuous in $\overline{D}$. The equicontinuity must be understood in the sense of the chordal metric h defined in (4.1).

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