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ON CARATHÉODORY THEOREM FOR ONE CLASS OF MAPPINGS

This article is devoted to the study of mappings of Orlicz-Sobolev classes that are not closed. In other words, we study the case when the mappings are open, discrete, but do not preserve the boundary of a domain (i.e., the corresponding cluster set of the mapping may contain inner points of the mapped domain). We investigated the problem of the existence of a limit of the above mapping at a fixed boundary point. We have established that such mappings have a continuous boundary extension whenever the corresponding Orlicz function satisfies the Calderon condition and the inner dilatation of the mapping satisfies some constraints on its growth, written in terms of singular parameters. In this case, we also impose conditions on the definition domain of the mapping. In particular, the domain must be locally connected on the boundary and finitely connected with respect to the pre-image of the corresponding cluster set. This approach is primarily related to the fulfillment of modulus inequalities of the Poletsky type in Orlicz-Sobolev classes. The available facts about the fulfillment of such inequalities require that the mapping be a homeomorphism, or more generally, an open discrete and closed mapping. If we consider more general classes of mappings, we cannot directly use the apparatus of modulus. Nevertheless, the conditions on the cluster set guarantee that the study of a mapping in an arbitrary domain can be reduced to some subdomain in which the mapping is already closed. The work is structured as follows. In the first part, we formulate the relevant definitions necessary for the formulation of the main result and present this formulation. The second section is devoted to the modulus of families of paths and surfaces. Here we formulate, in particular, a statement about the connection of Orlicz-Sobolev classes with upper modulus inequalities of the Poletsky type. The third part of the work is the proof of the main result, Theorem 1. The proof is carried out by the method of contradiction. The main tool of the proof is the apparatus of paths. We construct paths in the pre-image under the mapping, the diameter of which tends to zero. The ends of the paths are elements of sequences of points converging to a given point of the boundary. If the mapping has no boundary, the diameter of the mapped paths does not tend to zero. We show that the latter contradicts both the geometry of the definition domain and the proposed estimates of the growth of the dilatations of the mappings.

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Key words: *boundary behavior, Orlicz-Sobolev classes, moduli families of paths, quasiconformal mappings*

1. Introduction. Recall some definitions. Let U be an open set in $\mathbb{R}^n$. In what follows, $C_0^k(U)$ denotes the space of functions $u : U \to \mathbb{R}$ with a compact support in U, having k partial derivatives with respect to any variable that are continuous in U. Let U be an open set, $U \subset \mathbb{R}^n$, and let $u : U \to \mathbb{R}$ be some function, $u \in L^1_{\rm loc}(U)$. Suppose that, there is a function $v \in L^1_{\rm loc}(U)$ such that

$$\int\limits_U \frac{\partial \varphi}{\partial x_i}(x)u(x)\,dm(x) = -\int\limits_U \varphi(x)v(x)\,dm(x)$$

for any function $\varphi \in C_1^0(U)$. Then we say that the function v is a weak derivative of the first order of the function u with respect to x_i and denote it by $\frac{\partial u}{\partial x_i}(x) := v$.

A function $u \in W_{\rm loc}^{1,1}(U)$ if u has weak derivatives of the first order with respect to each of the variables in U, which are locally integrable in U. A mapping $f : U \to \mathbb{R}^n$ belongs to the Sobolev class $W_{\rm loc}^{1,1}(U)$, write $f \in W_{\rm loc}^{1,1}(U)$, if all coordinate functions of $f = (f_1, \ldots, f_n)$ have weak partial derivatives of the first order, which are locally integrable in U. We write $f \in W_{\rm loc}^{1,p}(U)$, $p \geqslant 1$, if all coordinate functions of $f = (f_1, \ldots, f_n)$ have weak partial derivatives of the first order, which are locally integrable in U to the degree p.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function. The *Orlicz-Sobolev class* $W_{\rm loc}^{1,\varphi}(D)$ is the class of all locally integrable functions f with the first weak derivatives whose gradient ∇f belongs locally in D to the Orlicz class. Note that by the definition $W_{\rm loc}^{1,\varphi} \subset W_{\rm loc}^{1,1}$. For the case when $\varphi(t) = t^p$, $p \geqslant 1$, we write as usual $f \in W_{\rm loc}^{1,p}$.

Later on, we also write $f \in W_{\rm loc}^{1,\varphi}(D)$ for a locally integrable vector-function $f = (f_1, \ldots, f_m)$ of n real variables $x_1, \ldots, x_n$ if $f_i \in W_{\rm loc}^{1,1}(D)$ and

$$\int\limits_G \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty$$

for any domain $G \subset D$ such that $\overline{G} \subset D$, where $|\nabla f(x)| = \sqrt{\sum\limits_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}$.

Recall that a mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of each point $y \in \mathbb{R}^n$ consists of isolated points, and *is open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$.

The boundary of D is called *weakly flat* at the point $x_0 \in \partial D$, if for every $P > 0$ and for any neighborhood U of the point x_0 there is a neighborhood $V \subset U$ of the same point such that $M(\Gamma(E, F, D)) > P$ for any continua $E, F \subset D$ such that $E \cap \partial U \neq \varnothing \neq E \cap \partial V$ and $F \cap \partial U \neq \varnothing \neq F \cap \partial V$. The boundary of D is called weakly flat if the corresponding property holds at any point of the boundary of a domain D. Given a mapping $f : D \to \mathbb{R}^n$, we denote

$$C(f, x) := \{y \in \overline{\mathbb{R}^n} : \exists\, x_k \in D : x_k \to x, f(x_k) \to y, k \to \infty\} \tag{1}$$

and

$$C(f, \partial D) = \bigcup\limits_{x \in \partial D} C(f, x)\,. \tag{2}$$

In what follows, ${\rm Int}\, A$ denotes the set of inner points of the set $A \subset \overline{\mathbb{R}^n}$. Recall that, the set $U \subset \overline{\mathbb{R}^n}$ is neighborhood of the point z_0, if $z_0 \in {\rm Int}\, A$.

We say that the boundary ∂D of a domain D in $\mathbb{R}^n$, $n \geqslant 2$, is *strongly accessible at a point $x_0 \in \partial D$ with respect to the p-modulus* if for each neighborhood U of x_0 there exist a compact set $E\subset D$, a neighborhood $V\subset U$ of x_0 and $\delta>0$ such that

$$M_p(\Gamma(E,F,D))\geqslant\delta \tag{3}$$

for each continuum F in D that intersects ∂U and ∂V. When $p=n$, we will usually drop the prefix in the "p-modulus" when speaking about (3).

Assume that, a mapping f has partial derivatives almost everywhere in D. In this case, we set

$$\begin{aligned} l\left(f^{\,\prime}(x)\right) &= \min\limits_{h\in\mathbb{R}^n\setminus\{0\}}\frac{|f^{\,\prime}(x)h|}{|h|}\,,\\ \|f^{\,\prime}(x)\| &= \max\limits_{h\in\mathbb{R}^n\setminus\{0\}}\frac{|f^{\,\prime}(x)h|}{|h|}\,,\\ J(x,f) &= \det f^{\,\prime}(x)\,. \end{aligned} \tag{4}$$

and define for any $x\in D$ and $\alpha\geqslant 1$

$$K_{I,\alpha}(x,f)\quad=\quad\begin{cases}\frac{|J(x,f)|}{l(f^{\,\prime}(x))^{\,\alpha}}, & J(x,f)\ne 0,\\ 1, & f^{\,\prime}(x)=0,\\ \infty, & \text{otherwise}\end{cases}\,, \tag{5}$$

The version of the following statement for ring Q-mappings was given in [1].

Theorem 1. *Let $\alpha\geqslant 1$, let D and $D^{\,\prime}$ be bounded domains in $\mathbb{R}^n$, $n\geqslant 3$, let $b\in\partial D$, let $Q:D\rightarrow[0,\infty]$ be a Lebesgue measurable function and let $\varphi:[0,\infty)\rightarrow[0,\infty)$ be an increasing function. Let $f:D\rightarrow D^{\,\prime}$ be a bounded open discrete mapping of the class $W_{\rm loc}^{1,\varphi}(D)$, $f(D)=D^{\,\prime}$. In addition, assume that $C(f,\partial D)\subset E_*$ for some closed (with respect to $D^{\,\prime}$) set $E_*\subset D^{\,\prime}$ and $f^{\,-1}(E_*)=E$ for some closed (with respect to D) set $E\subset D$, besides that,*

1) the set E is nowhere dense in D and D is finitely connected on $E\cup\partial D$, i.e., for any $z_0\in E\cup\partial D$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components,

2) for any neighborhood U of b there is a neighborhood $V\subset U$ of b such that:

2a) $V\cap D$ is connected,

2b) $(V\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$,

3) all components of the set $D^{\,\prime}\setminus E_*$ have a strongly accessible boundary with respect to α-modulus,

4) the function φ satisfies the following Calderon condition

$$\int\limits_1^{\infty}\left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}}\,dt<\infty\,. \tag{6}$$

Suppose that $K_{I,\alpha}(x,f) \leqslant Q(x)$ a.e. and the following condition holds: there is $\varepsilon_0=\varepsilon_0(b)>0$ and some positive measurable function $\psi:(0,\varepsilon_0)\to(0,\infty)$ such that

$$0<I(\varepsilon,\varepsilon_0)=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt<\infty \tag{7}$$

for sufficiently small $\varepsilon\in(0,\varepsilon_0)$ and, in addition,

$$\int\limits_{A(b,\varepsilon,\varepsilon_0)} Q(x)\cdot\psi^{\,\alpha}(|x-b|)\;dm(x)=o(I^{\alpha}(\varepsilon,\varepsilon_0))\,, \tag{8}$$

where $A:=A(b,\varepsilon,\varepsilon_0)$ is defined in (10). Then f has a continuous extension to b.

2. Preliminaries. A Borel function $\rho:\mathbb{R}^n\to[0,\infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma}\rho(x)\,|dx|\geqslant 1$$

holds for all (locally rectifiable) paths $\gamma\in\Gamma$. In this case, we write: $\rho\in\operatorname{adm}\Gamma$. Let $p\geqslant 1$, then *p-modulus* of Γ is defined by the equality

$$M_p(\Gamma)=\inf_{\rho\in\,\operatorname{adm}\Gamma}\int\limits_{\mathbb{R}^n}\rho^p(x)\,dm(x)\,.$$

Let $x_0\in\mathbb{R}^n$, $0<r_1<r_2<\infty$,

$$\begin{aligned} S(x_0,r)&=\{x\,\in\,\mathbb{R}^n:|x-x_0|=r\}\,,\\ B(x_0,r)&=\{x\,\in\,\mathbb{R}^n:|x-x_0|<r\}\end{aligned} \tag{9}$$

and

$$A=A(x_0,r_1,r_2)=\{x\,\in\,\mathbb{R}^n:r_1<|x-x_0|<r_2\}\;. \tag{10}$$

The most important tool of the paper is the connection between Sobolev (Orlicz-Sobolev) classes and lower (ring) Q-mappings. The theoretical part of this connection is mainly established in [2].

Let ω be an open set in $\overline{\mathbb{R}^k}:=\mathbb{R}^k\cup\{\infty\}$, $k=1,\ldots,n-1$. A (continuous) mapping $S:\omega\to\mathbb{R}^n$ is called a k-dimensional surface S in $\mathbb{R}^n$. Sometimes we call the image $S(\omega)\subseteq\mathbb{R}^n$ the surface S, too. The number of preimages

$$N(S,y)\;=\;\operatorname{card}\,S^{-1}(y)=\operatorname{card}\{x\in\omega:\;S(x)=y\} \tag{11}$$

is said to be a *multiplicity function* of the surface S at a point $y\in\mathbb{R}^n$. In other words, $N(S,y)$ denotes the multiplicity of covering of the point y by the surface S. It is known that the multiplicity function is lower semi continuous, i.e.,

$$N(S,y)\geqslant\liminf_{m\to\infty}:N(S,y_m)$$

for every sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \ldots$, such that $y_m \to y \in \mathbb{R}^n$ as $m \to \infty$, see, e.g., [3], p. 160. Thus, the function $N(S, y)$ is Borel measurable and hence measurable with respect to every Hausdorff measure $\mathcal{H}^k$ (see, e.g., [4], p. 52).

If $\rho : \mathbb{R}^n \to [0, \infty]$ is a Borel function, then its *integral over a k-dimensional surface* S in $\mathbb{R}^n$, $n \geqslant 2$, is defined by the equality

$$\int\limits_S \rho \, d\mathcal{A} \ := \int\limits_{\mathbb{R}^n} \rho(y) \, N(S, y) \, d\mathcal{H}^k y \ . \tag{12}$$

Given a family $\mathcal{S}$ of such k-dimensional surfaces S in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for $\mathcal{S}$, abbr. $\rho \in \operatorname{adm} \mathcal{S}$, if

$$\int\limits_S \rho^k \, d\mathcal{A} \ \geqslant \ 1 \tag{13}$$

for every $S \in \mathcal{S}$. Given $p \in [k, \infty)$, the *p-modulus* of $\mathcal{S}$ is the quantity

$$M_p(\mathcal{S}) \ = \inf_{\rho \in \operatorname{adm} \mathcal{S}} \int\limits_{\mathbb{R}^n} \rho^p(x) \, dm(x) \, . \tag{14}$$

The next class of mappings is a generalization of quasiconformal mappings in the sense of Gehring's ring definition (see [5]; cf. [6, Chapter 9]). Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 2$. Suppose that $x_0 \in \overline{D} \setminus \{\infty\}$ and $Q \colon D \to (0, \infty)$ is a Lebesgue measurable function. A metric ρ is said to be *extensively admissible* for Γ ($\rho \in \operatorname{ext}_p \operatorname{adm} \Gamma$) with respect to p-modulus if $\rho \in \operatorname{adm} (\Gamma \backslash \Gamma_0)$ such that $M_p(\Gamma_0) = 0$. A mapping $f : D \to D'$ is called a *lower Q-mapping at a point x_0 relative to p-modulus* if

$$M_p(f(\Sigma_\varepsilon)) \geqslant \inf_{\rho \in \operatorname{ext}_p \operatorname{adm} \Sigma_\varepsilon} \int\limits_{D \cap A(x_0, \varepsilon, r_0)} \frac{\rho^p(x)}{Q(x)} \, dm(x) \tag{15}$$

for every spherical ring $A(x_0, \varepsilon, r_0) = \{x \in \mathbb{R}^n \ : \ \varepsilon < |x - x_0| < r_0\}$, $r_0 \in (0, d_0)$, $d_0 = \sup\limits_{x \in D} |x - x_0|$, where Σ_ε is the family of all intersections of the spheres $S(x_0, r)$ with the domain D, $r \in (\varepsilon, r_0)$. If $p = n$, we say that f is a lower Q-mapping at x_0. We say that f is a lower Q-mapping relative to p-modulus in $A \subset \mathbb{R}^n$ if (15) is true for all $x_0 \in A$.

Let

$$N(y, f, A) \ = \ \operatorname{card} \{x \in A : f(x) = y\} \ , \tag{16}$$

$$N(f, A) \ = \ \sup_{y \in \mathbb{R}^n} N(y, f, A) \, . \tag{17}$$

The following lemma holds, see e.g. [7, Lemma 5.1].

Lemma 1. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $\varphi \colon (0, \infty) \to (0, \infty)$ be a monotone nondecreasing function satisfying (6). If $p > n - 1$, then every open discrete* *mapping $f\colon D\to\mathbb{R}^n$ of class $W^{1,\varphi}_{\rm loc}$ with finite distortion and such that $N(f,D)<\infty$ is a lower Q-mapping relative to the p-modulus at every point $x_0\in\overline{D}$ for*

$$Q(x)=N(f,D)\cdot K_{I,\alpha}^{\frac{p-n+1}{n-1}}(x,f),$$

$\alpha:=\frac{p}{p-n+1}$, where the inner dilation $K_{I,\alpha}(x,f)$ for f at x of order α is defined by (5), and the multiplicity $N(f,D)$ is defined by the relation (17).

Given sets E, $F\subset\overline{\mathbb{R}^n}$ and a domain $D\subset\mathbb{R}^n$ we denote by $\Gamma(E,F,D)$ a family of all paths $\gamma:[a,b]\to\overline{\mathbb{R}^n}$ such that $\gamma(a)\in E, \gamma(b)\in\ F$ and $\gamma(t)\in D$ for $t\in(a,b)$. Let $S_i=S(x_0,r_i)$, $i=1,2$, where spheres $S(x_0,r_i)$ centered at x_0 of the radius r_i are defined in (9). The following statement holds, see [2].

Lemma 2. *Let $x_0\in\partial D$, let $f:D\to\mathbb{R}^n$ be a bounded, open, discrete, and closed lower Q-mapping with respect to p-modulus in a domain $D\subset\mathbb{R}^n$, $Q\in L_{loc}^{\frac{n-1}{p-n+1}}(\mathbb{R}^n)$, $n-1<p$, and $\alpha:=\frac{p}{p-n+1}$. Then, for every $\varepsilon_0<d_0:=\sup\limits_{x\in D}|x-x_0|$ and every compact set $C_2\subset D\setminus B(x_0,\varepsilon_0)$ there exists ε_1, $0<\varepsilon_1<\varepsilon_0$, such that, for each $\varepsilon\in(0,\varepsilon_1)$ and each compact $C_1\subset\overline{B(x_0,\varepsilon)}\cap D$ the inequality*

$$M_{\alpha}(f(\Gamma(C_1,C_2,D)))\leqslant\int\limits_{A(x_0,\varepsilon,\varepsilon_1)}Q^{\frac{n-1}{p-n+1}}(x)\eta^{\alpha}(|x-x_0|)\,dm(x)\tag{18}$$

holds, where $A(x_0,\varepsilon,\varepsilon_1)=\{x\in\mathbb{R}^n:\varepsilon<|x-x_0|<\varepsilon_1\}$ and $\eta:(\varepsilon,\varepsilon_1)\to[0,\infty]$ is an arbitrary Lebesgue measurable function such that

$$\int\limits_{\varepsilon}^{\varepsilon_1}\eta(r)\,dr=1\,.\tag{19}$$

Let $D\subset\mathbb{R}^n$, let $f:D\to\mathbb{R}^n$ be a discrete open mapping, $\beta:[a,\,b)\to\mathbb{R}^n$ be a path, and $x\in\,f^{\,-1}(\beta(a))$. A path $\alpha:[a,\,c)\to D$ is called a *maximal f-lifting* of β starting at x, if (1) $\alpha(a)=x\,$; (2) $f\circ\alpha=\beta|_{[a,\,c)}$; (3) for $c<c'\leqslant b$, there is no a path $\alpha':[a,\,c')\to D$ such that $\alpha=\alpha'|_{[a,\,c)}$ and $f\circ\alpha'=\beta|_{[a,\,c')}$. Here and below we say that a path $\beta:[a,b)\to\overline{\mathbb{R}^n}$ converges to the set $C\subset\overline{\mathbb{R}^n}$ as $t\to b$, if $h(\beta(t),C)=\sup\limits_{x\in C}h(\beta(t),C)\to 0$ at $t\to b$. The following is true (see [8, Lemma 3.12]).

Proposition 1. Let $f:D\to\mathbb{R}^n$, $n\geqslant 2$, be an open discrete mapping, let $x_0\in D$, and let $\beta:[a,\,b)\to\mathbb{R}^n$ be a path such that $\beta(a)=f(x_0)$ and such that either $\lim\limits_{t\to b}\beta(t)$ exists, or $\beta(t)\to\partial f(D)$ as $t\to b$. Then β has a maximal f-lifting $\alpha:[a,\,c)\to D$ starting at x_0. If $\alpha(t)\to x_1\in D$ as $t\to c$, then $c=b$ and $f(x_1)=\lim\limits_{t\to b}\beta(t)$. Otherwise $\alpha(t)\to\partial D$ as $t\to c$.

The following statement is proved in [1, Lemma 2.1].

Lemma 3. *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $x_0 \in \partial D$. Assume that E is closed and nowhere dense in D, and D is finitely connected on E, i.e., for any $z_0 \in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of z_0 such that $(D \cap \widetilde{V}) \setminus E$ consists of finite number of components.*

In addition, assume that the following condition holds: for any neighborhood U of x_0 there is a neighborhood $V \subset U$ of x_0 such that:

a) $V \cap D$ is connected,

b) $(V \cap D) \setminus E$ consists at most of m components, $1 \leqslant m < \infty$.

Let $x_k, y_k \in D \setminus E$, $k = 1, 2, \ldots,$ be a sequences converging to x_0 as $k \to \infty$. Then there are subsequences x_{k_l} and y_{k_l}, $l = 1, 2, \ldots,$ belonging to some sequence of neighborhoods V_l, $l = 1, 2, \ldots,$ of the point x_0 such that $V_l \subset B(x_0, 2^{-l})$, $l = 1, 2, \ldots,$ and, in addition, any pair x_{k_l} and y_{k_l} may be joined by a path γ_l in $V_l \cap D$, where γ_l contains at most $m-1$ points in E.

3. Proof of Theorem 1. Firstly let us to prove that, under conditions of Theorem 1, $D \setminus E$ consists of finite number of components $D_1, \ldots, D_s$, $1 \leqslant s < \infty$. Indeed, assume the contrary, namely, that $D \setminus E$ consists of infinite number of components $D_1, D_2, \ldots$. Let $z_i \in D_i$, $i = 1, 2, \ldots$. Since D is bounded, there is a subsequence z_{i_k}, $k_1, 2, \ldots,$ which converge to some point $z_0 \in \overline{D}$. On the other hand, there is a neighborhood V of the point z_0 such that $(V \cap D) \setminus E$ consists of m components. Thus, there is at least one such a component K intersecting infinitely many components $D_{i_1}, D_{i_2}, \ldots$. However, by the assumption there is a neighborhood V of z_0 such that $(V \cap D) \setminus E$ consists of finite number of components $V_1, \ldots, V_m$. Now, there is $m_0 \in [1, m]$ such that V_{m_0} intersects infinitely many D_i, contradiction. Thus, $D_1, \ldots, D_s$, $1 \leqslant s < \infty$, as required.

Now, let us to prove that f is closed in each D_i, $i = 1, 2, \ldots,$ i.e. $f(S)$ is closed in $f(D_i)$ whenever S is closed in D_i. Since f is open and discrete, it is sufficient to prove that f is boundary preserving, that is $C(f, \partial D_i) \subset \partial f(D_i)$ (see, e.g., [9, Theorem 3.3]). Indeed, let $z_k \in D_i$, $k = 1, 2, \ldots,$ let $z_k \to z_0 \in \partial D_i$ and let $f(z_k) \to w_0$ as $k \to \infty$. We need to prove that $w_0 \in \partial f(D_i)$. Assume the contrary, i.e. $w_0 \in f(D_i)$. There are two cases: $z_0 \in \partial D$ or $z_0 \in D$. 1) In the first case, when $z_0 \in \partial D$, we obtain that $w_0 \in C(f, \partial D) \subset E_*$. But since $w_0 \in f(D_i)$, there is $\zeta_i \in D_i$ such that $f(\zeta_i) = w_0$. Now, since $w_0 \in E_*$, we obtain that $\zeta_i \in f^{-1}(E_*) = E$ and, consequently, $\zeta_i \not\in D_i$, contradiction. 2) Let us consider the second case, when $z_0 \in D$. Now, by the definition of D_i, we have that $z_0 \in E$. Now, $w_0 \in f(E) \subset E_*$. But since $w_0 \in f(D_i)$, there is $\zeta_i \in D_i$ such that $f(\zeta_i) = w_0$. Since $w_0 \in E_*$, we have that $\zeta_i \in f^{-1}(E_*) = E$ and, consequently, $\zeta_i \not\in D_i$, contradiction.

Let K be a component of $D' \setminus E_*$ consisting $f(D_i)$. Observe that $f(D_i) = K$. Indeed, $f(D_i) \subset K$ by the definition. Let us to prove that $K \subset f(D_i)$. Let us prove this inclusion by the contradiction, i.e., let $a_0 \in K \setminus f(D_i)$. Chose $b_0 \in f(D_i)$ and join the points b_0 and a_0 by a path $\beta : [0, 1] \to K$. Let $\alpha : [0, c) \to D$ be a maximal f-lifting of β starting at $c_0 := f^{-1}(b_0) \cap D_i$ (this lifting exists by Proposition 1). By the same proposition either one of two cases are possible: $\alpha(t) \to x_1$ as $t \to c$, or $\alpha(t) \to \partial D_i$ as

$t \to c$. In the first case, by Proposition 1 we obtain that $c = 1$ and $f(\beta(1)) = f(x_1) = a_0$ which contradict with the choice of a_0. In the second case, when $\alpha(t) \to \partial D_i$ as $t \to c$, we obtain that $f(\beta(c)) \in C(f, \partial D_i) \subset E_*$. The latter contradicts the definition of β because β does not contain itself points in E_*.

Observe that, $D' \setminus E_*$ consists of finite number of components. Otherwise, $D' \setminus E_* = \bigcup\limits_{i=1}^{\infty} K_i'$, $\varnothing \neq K_i' \neq K_j' \neq \varnothing$ for $i \neq j$. Let K_i be some component of the set $f^{-1}(K_i')$. By the definition $K_i \subset D \setminus E$ and $\varnothing \neq K_i \neq K_j \neq \varnothing$ for $i \neq j$. The latter contradicts the proving above for the number of components of $D \setminus E$.

Now, we proceed directly to the proof of the main statement of Theorem 1. We apply the approach used under the proof of [1, Lemma 3.1]. Suppose the opposite, i.e., the conclusion of the theorem is not true. Since D' is bounded, there are at least two sequences x_k, $y_k \in D$, $i = 1, 2, \ldots$, such that x_k, $y_k \in D$, $k = 1, 2, \ldots$, $x_k \to b$, $y_k \to b$ as $k \to \infty$, and $f(x_k) \to y$, $f(y_k) \to y'$ as $k \to \infty$, while $y' \neq y$. In particular,

$$|f(x_k) - f(y_k)| \geqslant \delta > 0 \tag{20}$$

for some $\delta > 0$ and all $k \in \mathbb{N}$. By the assumption 2), there exists a sequence of neighborhoods $V_k \subset B(b, 2^{-k})$, $k = 1, 2, \ldots$, such that $V_k \cap D$ is connected and $(V_k \cap D) \setminus E$ consists at most of m components, $1 \leqslant m < \infty$.

We note that the points x_k and y_k, $k = 1, 2, \ldots$, may be chosen such that $x_k, y_k \notin E$. Indeed, since under condition 1) the set E is nowhere dense in D, there exists a sequence $x_{ki} \in D \setminus E$, $i = 1, 2, \ldots$, such that $x_{ki} \to x_k$ as $i \to \infty$. Put $\varepsilon > 0$. Due to the continuity of the mapping f at the point x_k, for the number $k \in \mathbb{N}$ there is a number $i_k \in \mathbb{N}$ such that $|f(x_{ki_k}) - f(x_k)| < \frac{1}{2^k}$. So, by the triangle inequality

$$|f(x_{ki_k}) - y| \leqslant |f(x_{ki_k}) - f(x_k)| + |f(x_k) - y| \leqslant \frac{1}{2^k} + \varepsilon\,,$$

$k \geqslant k_0 = k_0(\varepsilon)$, because $f(x_k) \to y$ as $k \to \infty$ and by the choice of x_k and y. Therefore, $x_k \in D$ may be replaced by $x_{ki_k} \in D \setminus E$, as required. We may reason similarly for the sequence y_k.

Now, by Lemma 3 there are subsequences x_{k_l} and y_{k_l}, $l = 1, 2, \ldots$, belonging to some sequence of neighborhoods V_l, $l = 1, 2, \ldots$, of the point b such that $\operatorname{diam} V_l \to 0$ as $l \to \infty$ and, in addition, any pair x_{k_l} and y_{k_l} may be joined by a path γ_l in V_l, where γ_l contains at most $m-1$ points in E. Without loss of generality, we may assume that the same sequences x_k and y_k satisfy properties mentioned above. Let $\gamma_k : [0, 1] \to D$, $\gamma_k(0) = x_k$ and $\gamma_k(1) = y_k$, $k = 1, 2, \ldots$.

Observe that, the path $f(\gamma_k)$ contains not more than $m - 1$ points in E_*. In the contrary case, there are at least m such points $b_1 = f(\gamma_k(t_1)), b_2 = f(\gamma_k(t_2)), \ldots, b_m = f(\gamma_k(t_m))$, $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m \leqslant 1$. But now the points $a_1 = \gamma_k(t_1), a_2 = \gamma_k(t_2), \ldots, a_m = \gamma_k(t_m)$ are in $f^{-1}(E_*) = E$ and simultaneously belong to γ_k. This contradicts the definition of γ_k.

Let

$$b_1 = f(\gamma_k(t_1)), b_2 = f(\gamma_k(t_2)) \quad , \ldots , \quad b_l = f(\gamma_k(t_1))\,,$$
$$0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_l \leqslant 1, \qquad 1 \leqslant l \leqslant m-1\,,$$

be points in $f(\gamma_k) \cap E_*$. By the relation (20) and due to the triangle inequality,

$$\delta \leqslant |f(x_k) - f(y_k)| \leqslant \sum_{k=1}^{l-1} |f(\gamma_k(t_k)) - f(\gamma_k(t_{k+1}))|\,. \tag{21}$$

It follows from (21) that, there is $1 \leqslant l = l(k) \leqslant m-1$ such that such that

$$|f(\gamma_k(t_{l(k)})) - f(\gamma_k(t_{l(k)+1})| \geqslant \delta/l \geqslant \delta/(m-1)\,. \tag{22}$$

Observe that, the set $G_k := |f(\gamma_k)|_{(t_{l(k)}, t_{l(k)+1})}|$ belongs to $D' \setminus E_*$, because it does not contain any point in E_*.

Since $D' \backslash E_*$ consists only of finite components, there exists at least one a component of $D' \setminus E_*$, containing infinitely many components of G_k. Without loss of generality, going to a subsequence, if need, we may assume that all G_k belong to one component K of $D' \setminus E_*$.

Due to the compactness of $\overline{\mathbb{R}^n}$, we may assume that the sequence $w_k := f(\gamma_k(t_{l(k)}))$, $k = 1, 2, \ldots$, converges to some a point $w_0 \in \overline{D'}$. Let us to show that $w_0 \in C(f, \partial D) \subset E_*$. Indeed, there are two cases: either $w_k = f(\gamma_k(t_{l(k)})) \in E_*$ for infinitely many k, or $w_k \notin E_*$ for infinitely many $k \in \mathbb{N}$. In the first case, the inclusion $w_0 \in E_*$ is obvious, because E_* is closed. In the second case, obviously, $w_k = f(x_k)$, but this sequence converges to $y \in C(f, \partial D) \subset E_*$ by the assumption.

By the assumption, each component of the set $D' \setminus E_*$ has a strongly accessible boundary with respect to α-modulus. In this case, for any neighborhood U of the point $w_0 \in \partial K$ there is a compact set $C_0' \subset D'$, a neighborhood V of a point w_0, $V \subset U$, and a number $P > 0$ such that

$$M_\alpha(\Gamma(C_0', F, K)) \geqslant P > 0 \tag{23}$$

for any continua F, intersecting ∂U and ∂V. Choose a neighborhood U of w_0 with $d(U) < \delta/2(m-1)$, where δ is from (20). Let C_0' and V be a compact set and a neighborhood corresponding to w_0.

Observe that, G_k contains some a continuum $\widetilde{G}_k$ in K intersecting ∂U and ∂V for sufficiently large k. Indeed, by the construction of G_k, there is a sequence of points $a_{s,k} := f(\gamma_k(p_s)) \to w_k := f(\gamma_k(t_{l(k)}))$ as $s \to \infty$ and $b_{s,k} = f(\gamma_k(q_s)) \to f(\gamma_k(t_{l(k)+1}))$ as $s \to \infty$, where $t_{l(k)}) < p_s < q_s < \gamma_k(t_{l(k)+1}))$ and $a_{s,k}, b_{s,k} \in K$. By the triangle inequality and by (22)

$$|a_{s,k} - b_{s,k}| \geqslant |b_{s,k} - w_k| - |w_k - a_{s,k}|$$

$$\geqslant |f(\gamma_k(t_{l(k)+1})) - w_k| - |f(\gamma_k(t_{l(k)+1})) - b_{s,k}| - |w_k - a_{s,k}| \tag{24}$$

$$\geqslant \delta/(m-1) - |f(\gamma_k(t_{l(k)+1}) - b_{s,k})| - |w_k - a_{s,k}|\,.$$

Since $a_{s,k} := f(\gamma_k(p_s)) \to w_k := f(\gamma_k(t_{l(k)}))$ as $s \to \infty$ and $b_{s,k} = f(\gamma_k(q_s)) \to f(\gamma_k(t_{l(k)+1}))$ as $s \to \infty$, it follows from the latter inequality that there is $s = s(k) \in \mathbb{N}$ such that

$$|a_{s(k),k} - b_{s(k),k}| \geqslant \delta/2(m-1)\,. \tag{25}$$

Since V is open, there is some neighborhood U_k of w_k such $U_k \subset V$. Since $a_{s,k} \to w_k := f(\gamma_k(t_{l(k)}))$ as $s \to \infty$, we may assume that $a_{s(k),k} \in V$. Now, we set

$$\widetilde{G}_k := f(\gamma_k)|_{[p_{s(k)}, q_{s(k)}]}\,.$$

In other words, $\widetilde{G}_k$ is a part of the path $f(\gamma_k)$ between points $a_{s(k),k}$ and $b_{s(k),k}$. Let us to show that $\widetilde{G}_k$ intersects ∂U and ∂V. Indeed, by the mentioned above, $a_{s(k),k} \subset V$, so that $V \cap \widetilde{G}_k \neq \varnothing$. In particular, $U \cap \widetilde{G}_k \neq \varnothing$, because $V \subset U$. On the other hand, by (25) $d(\widetilde{G}_k) \geqslant \delta/2(m-1)$, however, $d(U) < \delta/2(m-1)$ by the choice of U. In particular, $d(V) < \delta/2(m-1)$. It follows from this that

$$(\overline{\mathbb{R}^n} \setminus U) \cap \widetilde{G}_k \neq \varnothing\,, \qquad (\overline{\mathbb{R}^n} \setminus V) \cap \widetilde{G}_k \neq \varnothing\,.$$

Now, by [10, Theorem 1.I.5, §46]

$$\partial U \cap \widetilde{G}_k \neq \varnothing\,, \partial V \cap \widetilde{G}_k \neq \varnothing\,,$$

as required. Now, by (23),

$$M_\alpha(\Gamma(\widetilde{G}_k, C_0', K)) \geqslant P > 0\,, \qquad k = 1, 2, \ldots\,. \tag{26}$$

Let us to show that, the relation (26) contradicts the definition of f and conditions (7)–(8). Indeed, let us denote by Γ_k the family of all half-open paths $\beta_k : [a, b) \to \overline{\mathbb{R}^n}$ such that $\beta_k(a) \in |f(\gamma_k)|$, $\beta_k(t) \in K$ for all $t \in [a, b)$ and, moreover, $\lim\limits_{t \to b-0} \beta_k(t) := B_k \in C_0'$. Obviously, by (26)

$$M_\alpha(\Gamma_k) = M_\alpha(\Gamma(\widetilde{G}_k, F, K)) \geqslant P > 0\,, \qquad k = 1, 2, \ldots\,. \tag{27}$$

Since by the proving above $D \setminus E$ consists of finite number of components, we may assume that all paths $\nabla_k := \gamma_k|_{[p_{s(k)}, q_{s(k)}]}$ belong to (some) one component D_1 of $D \setminus E$. By the proving above, f is closed in D_1 and $f(D_1) = K$. Consider the family Γ_k' of maximal f-liftings $\alpha_k : [a, c) \to D$ of the family Γ_k starting at $|\nabla_k|$; such a family exists by Proposition 1.

Note that, $C(f, \partial D_1) \subset C(f, \partial D) \cup C(f, E) \subset E_*$. Now, observe that, the situation when $\alpha_k \to \partial D_1$ as $k \to \infty$ is impossible because f is closed in D_1 and, consequently, f is boundary preserving (see [9, Theorem 3.3]). Therefore, by Proposition 1 $\alpha_k \to x_1 \in D$ as $t \to c - 0$, and $c = b$ and $f(\alpha_k(b)) = f(x_1)$. In other words, the f-lifting α_k is complete, i.e., $\alpha_k : [a, b] \to D_1$. Besides that, it follows from that $\alpha_k(b) \in f^{-1}(C_0') \cap D_1$.

Since $f(D_1)=K$, we obtain that $f(f^{-1}(C_0^{\prime})\cap D_1)=C_0^{\prime}$. In addition, $f^{-1}(C_0^{\prime})\cap D_1$ is a compactum in D_1, see e.g. [9, Theorem 3.3]. Thus, there is $r_0^{\prime}>0$ such that

$$d(f^{-1}(C_0^{\prime})\cap D_1, \partial D)\geqslant r_0>0\,. \tag{28}$$

Moreover, it follows from (28) that

$$f^{-1}(C_0^{\prime})\subset D\setminus B(b, r_0)\,. \tag{29}$$

We may consider that $r_0<\varepsilon_0$, where ε_0 is a number from the conditions of the theorem. Applying now Lemmas 1 and 2, we obtain for $\alpha:=\frac{p}{p-n+1}$ that

$$\begin{aligned} M_{\alpha}(f(\Gamma_k^{\prime})) &= M_{\alpha}(\Gamma(\widetilde{G}_k, C_0^{\prime}, K)) \\ &\leqslant (N(f,D_1))^{\frac{n-1}{p-n+1}} \int\limits_{A(b,2^{-k},r_0)} Q(x)\eta^{\alpha}(|x-b|)\,dm(x) \end{aligned} \tag{30}$$

for sufficiently large $k\in\mathbb{N}$ for any nonnegative Lebesgue measurable function η satisfying the relation (19) with $\varepsilon=2^{-k}$ and $\varepsilon_0=r_0$. It is worth noting that $N(f,D_1)<\infty$ because f is closed in D_1, see e.g. [11, Lemma 3.3]. Observe that the function

$$\eta(t)=\left\{\begin{array}{rr} \psi(t)/I(2^{-k}, r_0), & t\in(2^{-k}, r_0),\\ 0, & t\in\mathbb{R}\setminus(2^{-k}, r_0)\,, \end{array}\right.$$

where $I(2^{-k}, r_0)$ is defined in (7), satisfies (19). Therefore, by (30) and (7)–(8) we obtain that

$$M_{\alpha}(f(\Gamma_k^{\prime}))=M_{\alpha}(\Gamma(\widetilde{G}_k, C_0^{\prime}, K))\leqslant (N(f,D_1))^{\frac{n-1}{p-n+1}}\Delta(k)\,, \tag{31}$$

where $\Delta(k)\rightarrow 0$ as $k\rightarrow\infty$. The relations (31) together with (26) contradict each other, which proves the theorem.

Some questions concerning the study of Orlicz–Sobolev classes are discussed in [12].

4. Declarations.

Conflict of interest

The authors declare that there is no conflict of interest.

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Author contributions

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References

1. Desyatka, V., & Sevost'yanov, E. (2025). On boundary-non-preserving mappings with Poletsky inequality. *Canadian Mathematical Bulletin, 68*(3), 834–855. https://doi.org/10.4153/S0008439525000141
2. Sevost'yanov, E. (2016). On boundary behavior of mappings of Sobolev and Orlicz–Sobolev class. *arXiv preprint arXiv:1601.03762.* https://doi.org/10.48550/arXiv.1601.03762
3. Rado, T., & Reichelderfer, P. V. (1955). *Continuous transformations in analysis.* Springer.
4. Saks, S. (1964). *Theory of the integral.* Dover Publications.
5. Gehring, F. W. (1962). Rings and quasiconformal mappings in space. *Transactions of the American Mathematical Society, 103*, 353–393. https://doi.org/10.2307/1993834
6. Martio, O., Ryazanov, V., Srebro, U., & Yakubov, E. (2009). *Moduli in modern mapping theory.* Springer.
7. Sevost'yanov, E. (2023). *Mappings with direct and inverse Poletsky inequalities* (Developments in Mathematics, Vol. 78). Springer Nature Switzerland.
8. Martio, O., Rickman, S., & Väisälä, J. (1971). Topological and metric properties of quasiregular mappings. *Annales Academiae Scientiarum Fennicae. Series A I Mathematica, 488*, 1–31. https://doi.org/10.5186/aasfm.1971.488
9. Vuorinen, M. (1976). Exceptional sets and boundary behavior of quasiregular mappings in n-space. *Annales Academiae Scientiarum Fennicae. Series A I. Mathematica Dissertationes, 11*, 1–44.
10. Kuratowski, K. (1968). *Topology* (Vol. 2). Academic Press.
11. Martio, O., & Srebro, U. (1975). Periodic quasimeromorphic mappings. *Journal d'Analyse Mathématique, 28*, 20–40.
12. Salimov, R. R. (2017). Metric properties of Orlicz–Sobolev classes. *Journal of Mathematical Sciences, 220*(5), 633–642. https://doi.org/10.1007/s10958-016-3206-2

Вікторія Десятка, Аліна Галицька, Євген Севостьянов

Про теорему Каратеодорі для одного класу відображень.

Дана стаття присвячена дослідженню відображень класів Орлича-Соболєва, які не є замкненими. Іншими словами, ми досліджуємо випадок, коли відображення є відкритими, дискретними та не зберігають межу області (тобто, відповідна множина точок накопичення відображення може містити в собі внутрішні точки відображеної області). Нами досліджено питання про наявність границі у таких відображень у фіксованій точці межі. Нами встановлено, що такі відображення мають неперервне межове продовження, якщо відповідна функція Орлича задовольняє умову Кальдерона, а внутрішня дилатація відображення задовольняє деякі обмеження на зростання, записані в термінах сингулярних параметрів. При цьому, ми також накладаємо умови на область, в якій задане відображення. Зокрема, ця область має бути локально зв' язною на межі і одночасно скінченно зв'язною по відношенню до прообразу множини точок накопичення. Такий підхід пов'язаний передусім з виконанням у класах Орлича-Соболєва модульних нерівностей типу Полецького. Наявні факти про виконання таких нерівностей вимагають, щоб відображення було або гомеоморфізмом, або більш загально, відкритим дискретним і замкненим відображенням. Якщо ж ми розглядаємо більш загальні класи відображень, ми не можемо напряму скористатися апаратом модулів. Тим не менш, умови на множину точок накопичення гарантують можливість зведення дослідження відображень у довільній області до дослідження відображень, визначених в деякій підобласті, де відображення вже є замкненим. Робота побудована наступним чином. У першій частині ми формулюємо відповідні означення, необхідні для наведення основного результату, та наводимо це формулювання. Другий розділ присвячений модулю сімей кривих та поверхонь. Тут ми формулюємо, зокрема, твердження про зв'язок класів Орлича-Соболєва з верхніми модульними нерівностями типу Полецького. Третя частина роботи – це доведення основного результату, теореми 1. Доведення відбувається методом від супротивного. Основним інструментом доведення є апарат кривих. Ми будуємо криві у прообразі при відображенні, діаметр яких прямує до нуля. Кінці кривої – це елементи послідовностей точок, збіжних до даної точки межі. Якщо відображення не має границі, діаметр відображених кривих до нуля не прямує. Ми показуємо, що останнє суперечить як геометрії області визначення, так і оцінкам зростання дилатацій відображень.

Ключові слова: *межова поведінка, класи Орлича-Соболєва, модулі сімей кривих, квазіконформні відображення.*

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