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ON THE POSSIBILITY OF JOINING TWO PAIRS OF POINTS IN CONVEX DOMAINS USING PATH

This article is devoted to the possibility of joining of two pairs of points of a convex domains by curves. We are interested in the case when these curves are not farther from each other than the distance between their end points, possibly, up to some absolute multiplicative constant. We have obtained some upper and lower bounds for the modulus of families of paths joining curves mentioned above.

MSC: Primary 52A20, Secondary 31A15.

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1. Introduction.

This manuscript is devoted to the possibility of joining points by curves in a bounded convex domain, as well as estimates of the modulus of families of paths joining them. The definition and properties of the modulus of families of paths can be found, for example, in [1]. It is worth noting that the study of convex domains can be put on a par with the study of John domains, uniform domains (separately according to Näkki and Martio and Sarvas), collared domains as well as quasiextremal distance domains by Gehring and Martio, see, e.g., [2–4] and [5]. Observe that, the study of classes of domains mentioned above has applications to the mapping theory. In particular, since the modulus of families of paths in quasiextremal distance domains (abbr. QED-domains) may be explicitly estimated in terms of the Euclidean diameter, one can obtain explicit estimates of the distortion under mappings in such domains. Recall that, the indicated bounds for the modulus are also called Loewner-type estimates. Wherein, the modulus of a family of paths joining approaching continua has a logarithmic order of growth, see, e.g., [6, Lemma 7.38] or [7, relations (3.9)–(3.10)].

These circumstances are the motivation for our research in this manuscript. We deal exclusively with bounded convex domains and show how exactly their geometry is related to the construction of curves of a special kind. More precisely, in such domains we join two different points with some another two different points by means of non-intersecting curves, so that the diameters of these curves are not too small, and the distance between two of the points is comparable to the distance between the curves as a whole. Due to this, the choice of some admissible function for the family of paths joining these curves becomes obvious. We obtain some upper and lower bounds for the above modulus. (In the future, we plan to obtain the corresponding applications of these estimates in the mapping theory, which will be published separately). We emphasize that, the results established here have already been obtained in particular case, when a domain is the unit ball [8]. Concerning some applications of modulus inequalities in the mapping theory, see [9], cf. [10–11].

Recall, that a set C is *convex* if any pair of points $x, y \in C$ may be joined by some segment which belongs to C, as well. We define the Euclidean distance between sets and the Euclidean diameter by the formulae

$$d(A,B)=\inf\limits_{x\in A, y\in B}|x-y|\,,\quad d(A)=\sup\limits_{x,y\in A}|x-y|\,.$$

Sometimes we also write $\mathrm{dist}\,(A,B)$ instead $d(A,B)$ and $\mathrm{diam}\,E$ instead $d(E)$, as well. As usually, we set

$$B(x_0,r)=\{x\in\mathbb{R}^n: |x-x_0|<r\}\,,$$
$$S(x_0,r)=\{x\in\mathbb{R}^n: |x-x_0|=r\}\,.$$

2. Auxiliary lemma.

The next statement includes some geometric properties of curves (segments), which are located in domains of special kind. We assume, that this statement will be useful for our further research with special distortion of modulus of families of paths. In particular, we mean quasiconformal mappings and wider mappings with direct and inverse Poletsky inequality (see, [1, 9–11]).

The following lemma was proved in the case where the domain D' is the unit ball (see the proof of Theorem 1.1 in [8]). For an arbitrary convex domain, its proof is significantly more difficult, since the previous methodology relied significantly on the geometry of the ball.

Lemma 1. *Let D' be a bounded convex domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $E:=B(y_*,\delta_*/2)$ be a ball centered at the point $y_*\in D'$, where $\delta_*:=d(y_*,\partial D')$. Let $z_0\in\partial D'$. Then for any points $A, B\in B(z_0,\delta_*/8)\cap D'$ there are points $C, D\in\overline{B(y_*,\delta_*/2)}$, for which the segments $[A,C]$ and $[B,D]$ are such that*

$$\mathrm{dist}\,([A,C],[B,D])\geqslant C_0\cdot|A-B|\,,\qquad(1)$$

where $C_0>0$ is some constant depending only on δ_* and $d(D')$.

Proof. We put

$$\varepsilon_0:=|A-B|<\delta_0:=\delta_*/4\,.$$

Let us join the points A and y_* with the segment I. Due to the convexity of D', the segment I completely belongs to D'. Points A, y_* and B form the plane P (if A, y_* and B belong to some straight line, then we define by P any plane, consisting A and B). Consider the circle on this plane

$$S=\{z\in P: |z-A|=\varepsilon_0\}\,.$$

The position of the point $z=B$ on the circle S is completely determined by the angle φ, $\varphi\in[-\pi,\pi)$ between the vector y_*-A and the radius vector $B-A$ of the point z.

Points on the circle are further denoted by polar coordinates using pairs $z = (\varepsilon_0, \varphi)$. Our the further goal is to investigate the main three cases regarding the intervals of change of this angle.

Case 1. "Large angles": $\varphi \in [\pi/4, 3\pi/4]$, or $\varphi \in [-\pi/4, -3\pi/4]$, see Figure 1 for illustrations. Put $C := y_*$. Let us limit ourselves to considering the case $\varphi \in [\pi/4, 3\pi/4]$

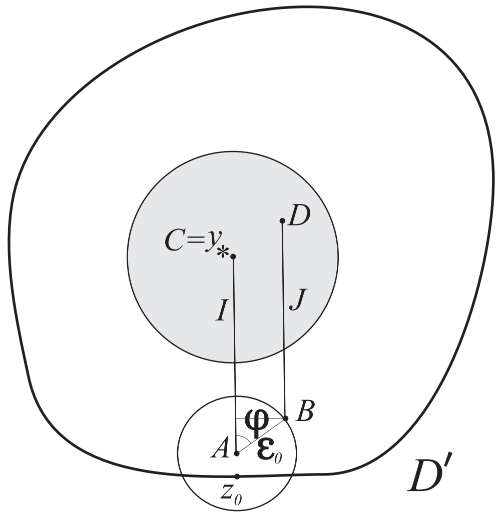

Figure 1. To proof of **Lemma 1**, *case* **1**

(a case $\varphi \in [-\pi/4, -3\pi/4]$ may be considered similarly). Consider the ray

$$r = r(t) = B + te\,, \quad e = (y_* - A)/|y_* - A|\,, \quad t > 0.$$

By construction, the ray r is parallel to the segment I. When $t = |y_* - A|$, we have $r(|y_* - A|) = B + y_* - A$ and $|r(|y_* - A|) - y_*| = |B - A| = \varepsilon_0 < \delta_0$, that is, point $r(|y_* - A|)$ belongs to E. Let J be a segment of the ray r, which is contained between the points B and $D := r(|y_* - A|)$ (it belongs entirely to D' due to its convexity). The distance between I and J is calculated as follows:

$$\mathrm{dist}\,(I, J) = \varepsilon_0 \sin\varphi \geqslant \frac{\sqrt{2}}{2}\varepsilon_0\,. \tag{2}$$

The consideration of the first case is completed, since we take I and J as segments $[A, C]$ and $[B, D]$, in addition, we put $C_0 := \frac{\sqrt{2}}{2}$ in (1).

Case 2. "Small angles": $\varphi \in [-\pi/4, \pi/4)$. Consider the case $\varphi \in [0, \pi/4)$ (the case $\varphi \in [-\pi/4, 0)$ may be considered similarly). Let us draw a line through the point y_*, which belongs to the plane P and is orthogonal to the vector $y_* - A$, see Figure 2. This line has exactly two points of intersection with the boundary of the circle $\overline{B(y_*, \delta_*/2)} \cap P$. We denote these points by x_{**} and z_{**}. Let x_{**} be a point such that the angle between by vectors $y_* - A$ and $x_{**} - A$ is negative. Let us put

$$x_* = \frac{x_{**} + y_*}{2}\,, \qquad z_* = \frac{z_{**} + y_*}{2}\,.$$

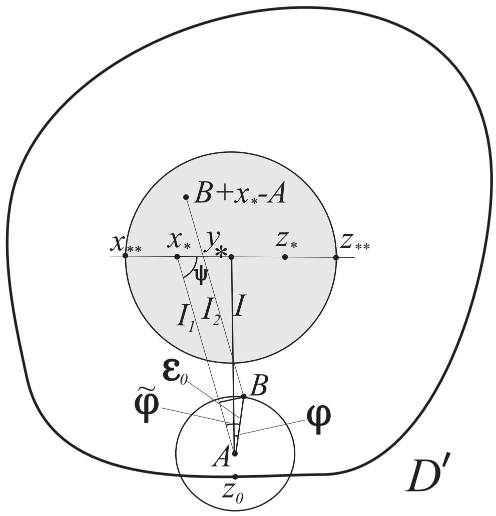

Figure 2. To proof of **Lemma 1**, *case* ***2***

Furthermore, let ψ denote the angle $\angle Ax_*y_*$. Then

$$\tan\psi=\frac{|A-y_*|}{\frac{\delta_*}{4}}<\frac{4d(D')}{\delta_*}\,. \tag{3}$$

In addition,

$$\tan\psi=\frac{|A-y_*|}{\frac{\delta_*}{4}}\geqslant\frac{\frac{\delta_*}{2}}{\frac{\delta_*}{4}}=2\,,$$

from here

$$\psi\geqslant\arctan 2>\frac{\pi}{4}\,,\qquad \frac{\pi}{2}-\psi\leqslant\frac{\pi}{2}-\arctan 2<\frac{\pi}{4}\,. \tag{4}$$

Accordingly, the angle $\angle y_*Ax_*$ is equal to $\frac{\pi}{2}-\psi$. Let

$$\widetilde{\varphi}:=\varphi+\frac{\pi}{2}-\psi\,,$$

where ψ is defined as above. Due to (4), since $\varphi\in[0,\pi/4)$,

$$0<\widetilde{\varphi}\leqslant\frac{\pi}{4}+\frac{\pi}{2}-\arctan 2=\frac{3\pi}{4}-\arctan 2<\frac{\pi}{2}\,. \tag{5}$$

Consider the ray $r(t)=B+t\frac{x_*-A}{|x_*-A|}$, $t>0$. If $t=|x_*-A|$, then

$$r(|x_*-A|)=B+x_*-A\,.$$

Note that, $r(|x_*-A|)\in\overline{B\left(y_*,\frac{\delta_*}{2}\right)}$. Indeed, by the triangle inequality,

$$|r(|x_*-A|)-y_*|=$$

$$= \big|B + x_* - A - y_*\big| \leqslant |x_* - y_*| + |B - A| \leqslant \frac{\delta_*}{4} + \frac{\delta_*}{4} = \frac{\delta_*}{2}\,. \tag{6}$$

Now let I_1 be the segment joining the points A and $C := x_*$, and let I_2 be the segment joining the points B and $D := r(|x_* - A|) = B + x_* - A$. By the construction and due to (6),

$$I_1 \cap \overline{B\left(y_*, \frac{\delta_*}{2}\right)} \neq \varnothing \neq I_2 \cap \overline{B\left(y_*, \frac{\delta_*}{2}\right)}\,. \tag{7}$$

Note that

$$\mathrm{dist}\,(I_1, I_2) = \varepsilon_0 \sin \widetilde{\varphi}\,. \tag{8}$$

Given the formulae (3) and (5), we obtain that

$$\tan \widetilde{\varphi} = \tan\left(\varphi + \frac{\pi}{2} - \psi\right) \geqslant$$

$$\geqslant \tan\left(\frac{\pi}{2} - \psi\right) = \cot \psi \geqslant \frac{\delta_*}{4d(D')}\,. \tag{9}$$

Due to (9) and taking into account (5), we obtain that

$$\sin \widetilde{\varphi} = \tan \widetilde{\varphi} \cdot \cos \widetilde{\varphi} \geqslant \frac{\delta_*}{4d(D')} \cdot \cos\left(\frac{3\pi}{4} - \arctan 2\right) > 0\,. \tag{10}$$

Now, by (8) and (10), we obtain that

$$\mathrm{dist}\,(I_1, I_2) \geqslant \varepsilon_0 \cdot \frac{\delta_*}{4d(D')} \cdot \cos\left(\frac{3\pi}{4} - \arctan 2\right) > 0\,. \tag{11}$$

As we have already noted, the case $\varphi \in [-\pi/4, 0)$ may be considered similarly, namely, the construction of the proof is the same “up to symmetry” with respect to the segment joining the points A and y_*. In particular, the role of the point x_* will be played by the point z_*; the role of the segment joining the points A and x_* preforms the segment joining the points A and z_*, etc.

The inequality (11) completes the consideration of case **2**, because we may put $[A, C] := I_1$ and $[B, D] := I_2$, in addition, we set $C_0 := \frac{\delta_*}{4d(D')} \cdot \cos\left(\frac{3\pi}{4} - \arctan 2\right)$ in the inequality (1).

Case 3. “Very large angles:” or $\varphi \in (3\pi/4, \pi]$, or $\varphi \in (-\pi, -3\pi/4)$. This case is very similar in case **2** at its discretion. It is enough to consider the situation $\varphi \in (3\pi/4, \pi]$, because the second situation $\varphi \in (-\pi, -3\pi/4)$ is considered similarly and differs from the first one only by a certain “symmetry” relative to the line passing through the points A and y_*. Let the points z_* and z_{**} are exactly as defined in case **2**. Consider the segment J_1, joining the points A and $C := z_*$ (it completely belongs to D' due to its convexity). Consider a ray

$$l(t) = B + t \cdot \frac{z_* - A}{|z_* - A|}, \qquad t > 0\,. \tag{12}$$

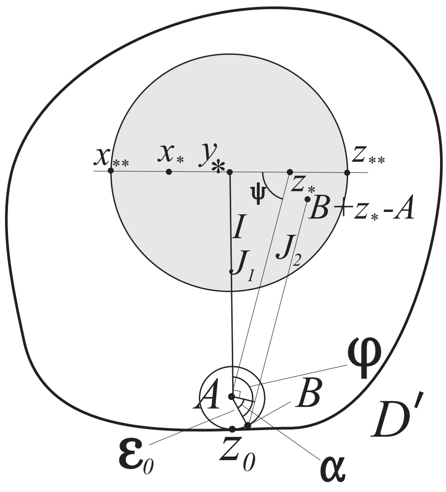

Figure 3. To proof of **Lemma 1**, *case* ***3***

For $t = |z_* - A|$, we obtain the point $D := l(|z_* - A|) = B + z_* - A$, see Figure 3. Let us prove that this point belongs to the ball $\overline{B\left(y_*, \frac{\delta_*}{2}\right)}$. Indeed, by the triangle inequality

$$|l(|z_* - A|) - y_*| =$$

$$= \left|B + z_* - A - y_*\right| \leqslant |z_* - y_*| + |B - A| \leqslant \frac{\delta_*}{4} + \frac{\delta_*}{4} = \frac{\delta_*}{2}\,. \tag{13}$$

Let J_2 be the segment joining the points B and $D := B + z_* - A$. By the construction and due to (13),

$$J_1 \cap \overline{B\left(y_*, \frac{\delta_*}{2}\right)} \neq \varnothing \neq J_2 \cap \overline{B\left(y_*, \frac{\delta_*}{2}\right)}\,. \tag{14}$$

Note that

$$\operatorname{dist}(J_1, J_2) = \varepsilon_0 \cos\alpha\,, \tag{15}$$

where

$$\alpha = \varphi - \frac{\pi}{2} - \left(\frac{\pi}{2} - \psi\right) = \varphi + \psi - \pi$$

and $\psi = \angle y_* z_* A$. Now we need to find the lower bound for $\cos\alpha$.

From the triangle $\triangle\, y_* z_* A$ we obtain that

$$1 \leqslant \tan\psi = \frac{|A - y_*|}{|y_* - z_*|} \leqslant \frac{d(D')}{\frac{\delta_*}{4}} = \frac{4d(D')}{\delta_*}\,,$$

from here $\frac{\pi}{4} \leqslant \psi \leqslant \arctan\frac{4d(D')}{\delta_*} < \frac{\pi}{2}$. Then

$$0 = \frac{3\pi}{4} + \frac{\pi}{4} - \pi \leqslant \alpha := \varphi + \psi - \pi \leqslant \pi + \arctan\frac{4d(D')}{\delta_*} - \pi = \arctan\frac{4d(D')}{\delta_*} < \frac{\pi}{2}\,. \tag{16}$$

By (15) and (16), we obtain that

$$\operatorname{dist}(J_1, J_2) \geq \varepsilon_0 \cos\arctan\frac{4d(D')}{\delta_*}\,. \tag{17}$$

Actually, inequality (17) completes the consideration of case **3**, since we may put $[A, C] := J_1$ and $[B, D] := J_2$, in addition, we put $C_0 := \cos\arctan\frac{4d(D')}{\delta_*}$ in (1).

Finally, for all cases **1**, **2** and **3** in (1) we may put

$$C_0 = \min\left\{\frac{\sqrt{2}}{2}, \frac{\delta_*}{4d(D')}\cdot\cos\left(\frac{3\pi}{4} - \arctan 2\right), \cos\arctan\frac{4d(D')}{\delta_*}\right\}.$$

□

3. Main results.

Recall that, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *an admissible* for a family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma} \rho(x)\,|dx| \geqslant 1 \tag{18}$$

holds for any locally rectifiable path $\gamma \in \Gamma$. *A modulus* of Γ is defined as follows:

$$M(\Gamma) = \inf_{\rho\in\,\mathrm{adm}\,\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\,dm(x)\,. \tag{19}$$

Theorem 1. *Let, under conditions of Lemma 1, Γ denotes the family of all paths joining the segments $[A, C]$ and $[B, D]$ in D'. Then*

$$M(\Gamma) \leqslant \frac{m(D')}{C_0^n}\cdot\frac{1}{|A - B|^n}\,, \tag{20}$$

where M is the modulus of families of paths defined in (19), $m(D')$ denotes the Lebesgue measure of D', and C_0 is a constant in (1).

Proof. Let us to set

$$\rho(x) = \begin{cases} C_0^{-1}\cdot|A - B|^{-1}, & x \in D', \\ 0, & x \notin D' \end{cases},$$

where C_0 is a constant in (1). Due to Lemma 1, ρ satisfies the relation (18). Then, the relation (20) directly follows by substituting ρ in the integral in (19) and by the definition of inf in it. □

In accordance with [5], a domain D in $\mathbb{R}^n$ is called a *domain of quasiextremal length* (abbr. *QED-domain*) if

$$M(\Gamma(E, F, \mathbb{R}^n)) \leqslant A_0^* \cdot M(\Gamma(E, F, D)) \tag{21}$$

for some finite number $A_0^*\geqslant 1$ and all continua E and F in D.

Theorem 2. *Let, under conditions of Lemma 1,* Γ *denotes the family of all paths joining the segments* $[A,C]$ *and* $[B,D]$ *in* D'. *Then*

$$M(\Gamma)\geqslant \widetilde{c_n}\cdot\log\left(1+\frac{3\delta_*}{8|A-B|}\right)\,, \tag{22}$$

where M *is the modulus of families of paths defined in (19),* $\widetilde{c_n}>0$ *is some constant depending only on* n *and* D'.

Proof. By [6, Lemma 7.38],

$$M(\Gamma([A,C],[B,D],\mathbb{R}^n))\geqslant c_n\cdot\log\left(1+\frac{1}{m}\right)\,, \tag{23}$$

where $c_n>0$ is some constant that depends only on n,

$$m=\frac{{\rm dist}([A,C],[B,D])}{\min\{{\rm diam}\,([A,C]),{\rm diam}\,([B,D])\}}\,.$$

Observe that, by the triangle inequality,

$$\begin{aligned}{\rm diam}\,([A,C])\geqslant |A-C|=|A-z_0+z_0-y_*+y_*-C|=\\ =|(z_0-y_*)+(A-z_0)+(y_*-C)|\geqslant\\ \geqslant|z_0-y_*|-|A-z_0|-|y_*-C|\geqslant\delta_*-\delta_*/8-\delta_*/2=3\delta_*/8\,.\end{aligned}$$

Similarly, we may prove that

$${\rm diam}\,([B,D])\geqslant 3\delta_*/8\,.$$

Now, the relation (23) implies that

$$M(\Gamma([A,C],[B,D],\mathbb{R}^n))\geqslant c_n\cdot\log\left(1+\frac{3\delta_*}{8|A-B|}\right)\,. \tag{24}$$

Observe that, D' is a John domain (see [2, Remark 2.4]). Thus, D' is a uniform domain (see [2, Remark 2.13(c)]). Thus, D' is a QED-domain with some $A_0^*<\infty$ in (21) (see [5, Lemma 2.18]). Due to (24) and (21), we obtain that

$$M(\Gamma([A,C],[B,D],D'))\geqslant \widetilde{c_n}\cdot\log\left(1+\frac{3\delta_*}{8|A-B|}\right)\,, \tag{25}$$

where $\widetilde{c_n}:=c_n/A_0^*$. The relation (22) has been proved. $\Box$

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О. Довгопятий

Про можливість з'єднання кривими двох пар точок в опуклих областях.

Дана стаття присвячена можливості з'єднання кривими двох пар точок опуклої області, які лежать одна від одної не далі, ніж відстань між їх кінцевими точками. В цьому контексті ми допускаємо ситуацію, коли відстані між кривими і точками можуть співвідноситися між собою з точністю до деякої абсолютної мультиплікативної сталої. Вказане питання піднімається нами з приводу пошуку відповідної допустимої функції сімей шляхів, що з'єднують ці криві. Наявність допустимої функції дозволяє оцінювати модуль сімей цих шляхів. По-перше, оцінку модуля зверху ми отримаємо тривіально. По-друге, ми стаємо спроможні доводити оцінки типу Льовнера у відповідних областях і, тим самим, мати явну оцінку евклідової відстані між кінцевими точками кривих через конформний модуль та їх діаметри. В свою чергу, така оцінка є найважливішою структурною складовою при дослідженні відображень з прямою і оберненою нерівностями Полецького, які діють з довільних областей на опуклі області. Вони дозволяють, зокрема, отримувати оцінки спотворення відстані при відображенні, у тому числі, неперервність за Гельдером, або (більш слабку) логарифмічну неперервність за Гельдером. Основною методологією, яка задіяна в статті, є плоский аналіз. Беручи дві точки, які належать до області евклідового простору, ми фіксуємо ще одну (третю) точку і робимо розріз за допомогою площини, яка містить їх усіх. Подальша конструкція будується переважно з урахуванням того, яким є кут між відрізками з початком в одній з точок та кінцем у кожній з інших точок. Відносно «легким» є випадок «середніх кутів», тобто, коли кут відділений від 0 та π з обох боків. Розгляд інших випадків є значно важчим методологічно. Фінальним етапом розгляду є синтез усіх трьох випадків в одну оцінку відстані між кривими із сталою, загальною для всіх згаданих ситуацій у цілому. Відзначимо, що області зі спеціальною геометрією розглядаються в теорії відображень доволі часто. До них можна віднести області квазіекстремальної довжини по Герінгу–Маріто, рівномірні області по Мартіо–Сарвасу, рівномірні області в сенсі Няккі, області Джона тощо. Слід зауважити, що обмежені опуклі області, які розглядаються в статті, є окремими випадком областей квазіекстремальної довжини. Чи є вірними результати статті для довільної області квазіекстремальної довжини, або до класів областей перелічених вище, наразі невідомо. На даний момент результати статті відомі лише для одиничної кулі; для довільних опуклих обмежених областей вони якраз встановлені в даній роботі.

Ключові слова: *опуклі області, евклідові простори, оцінки типу Льовнера, криві, модулі сімей кривих.*

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