UDC: 517.5
DOI: 10.37069/1683-4720-2021-35-8

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# ON MAPPINGS WITH ASYMPTOTIC HOMOGENEITY AT INFINITY AND BELTRAMI EQUATIONS

We establish here some criteria for the existence of regular homeomorphic solutions of the degenerate Beltrami equations in the complex plane with asymptotic homogeneity at infinity.

**MSC:** Primary 30C62, Secondary 30C65.

***Keywords:*** *degenerate Beltrami equations, conformality by Belinskij and conformality by Lavrent'iev, asymptotic homogeneity at infinity.*

*Dedicated to the memory of Professor Pavel Petrovich Belinskij (1928–1986)*

## 1. Introduction.

Let $D$ be a domain in the complex plane $\mathbb{C}$, i.e., a connected open subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in $D$. A **Beltrami equation** is an equation of the form

$$f_{\bar{z}} = \mu(z)\, f_z \tag{1}$$

with the formal complex derivatives $f_{\bar{z}} = \overline{\partial} f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, where $f_x$ and $f_y$ are partial derivatives of $f$ in $x$ and $y$, correspondingly. The function $\mu$ is said to be the **complex coefficient** and

$$K_\mu(z) \ := \ \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{2}$$

the **dilatation quotient** of the equation (1). The Beltrami equation is called **degenerate** if $\operatorname{ess\,sup} K_\mu(z) = \infty$. Homeomorphic solutions of the Beltrami equations with $K_\mu \le Q < \infty$ in the Sobolev class $W^{1,1}_{\rm loc}$ are called **Q-quasiconformal mappings**.

It is well known that if $K_\mu$ is bounded, then the Beltrami equation has homeomorphic solutions, see e.g. [1, 5, 24] and [45]. Recently, a series of effective criteria for the existence of homeomorphic $W^{1,1}_{\rm loc}$ solutions have been also established for degenerate Beltrami equations, see e.g. historic comments with relevant references in monographs the [3, 16] and [25], in BMO-article [36] and in the surveys [17] and [44].

These criteria were formulated both in terms of $K_\mu$ and the more refined quantity that takes into account not only the modulus of the complex coefficient $\mu$ but also its argument

$$K^T_\mu(z, z_0) \ := \ \frac{\left|1 - \frac{\overline{z-z_0}}{z-z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \tag{3}$$

that is called the **tangent dilatation quotient** of the Beltrami equation with respect to a point $z_0 \in \mathbb{C}$, see e.g. [2, 7, 8, 13, 23] and [36–41]. Note that

$$K_\mu^{-1}(z) \leqslant K_\mu^T(z, z_0) \leqslant K_\mu(z) \qquad \forall\, z \in D\,,\ z_0 \in \mathbb{C}\,. \tag{4}$$

The geometrical sense of $K_\mu^T$ can be found e.g. in the monographs [16] and [25].

By the well-known Gehring–Lehto–Menchoff theorem, see [12] and [28], see also the monographs [1] and [24], each homeomorphic $W^{1,1}_{\rm loc}$ solution $f$ of the Beltrami equation is differentiable a.e. Recall that a function $f : D \to \mathbb{C}$ is **differentiable by Darboux–Stolz at a point** $z_0 \in D$ if

$$f(z) - f(z_0) = f_z(z_0)(z - z_0) + f_{\overline{z}}(z_0)\overline{(z - z_0)} + o(|z - z_0|) \tag{5}$$

where $o(|z - z_0|)/|z - z_0| \to 0$ as $z \to z_0$. Moreover, $f$ is called **conformal at the point** $z_0$ if in addition $f_{\overline{z}}(z_0) = 0$ but $f_z(z_0) \neq 0$.

The example $w = z(1 - \ln|z|)$ of B.V. Shabat, see [4, p. 40], shows that, for a continuous complex characteristic $\mu(z)$, the quasiconformal mapping $w = f(z)$ can be non-differentiable by Darboux–Stolz at the origin. If the characteristic $\mu(z)$ is continuous at a point $z_0 \in D$, then, as was first established, apparently, by P.P. Belinskij in [4, p. 41], the mapping $w = f(z)$ is differentiable at $z_0$ in the following meaning:

$$\Delta w = A(\rho)\left[\Delta z + \mu_0\overline{\Delta z} + o(\rho)\right], \tag{6}$$

where $\mu_0 = \mu(z_0)$, $\rho = |\Delta z + \mu_0\overline{\Delta z}|$, $A(\rho)$ depends only on $\rho$ and $o(\rho)/\rho \to 0$ as $\rho \to 0$. As it was clarified later in [34], see also [14], here $A(\rho)$ may not have a limit with $\rho \to 0$, however,

$$\lim_{\rho\to 0}\frac{A(t\rho)}{A(\rho)} = 1 \quad \forall\, t > 0. \tag{7}$$

Following [34], a mapping $f : D \to \mathbb{C}$ is called **differentiable by Belinskij at a point** $z_0 \in D$ if conditions (6) and (7) hold with some $\mu_0 \in \mathbb{D} := \{\mu \in \mathbb{C} : |\mu| < 1\}$. Note that here, in the case of discontinuous $\mu(z)$, it is not necessary $\mu_0 = \mu(z_0)$. If in addition $\mu_0 = 0$, then $f$ is called **conformal by Belinskij at the point** $z_0$.

For quasiconformal mappings $f : D \to \mathbb{C}$ with $f(0) = 0 \in D$, it was shown in [34], see also [14], that the conformality by Belinskij of $f$ at the origin is equivalent to each of its properties:

$$\boxed{\lim_{\tau\to 0}\frac{f(\tau\zeta)}{f(\tau)} = \zeta} \qquad \text{along the ray } \tau > 0 \qquad \forall\, \zeta \in \mathbb{C}, \tag{8}$$

$$\boxed{\lim_{z\to 0}\left\{\frac{f(z')}{f(z)} - \frac{z'}{z}\right\} = 0} \qquad \text{along } z, z' \in \mathbb{C},\ |z'| < \delta|z|, \qquad \forall\, \delta > 0, \tag{9}$$

$$\boxed{\lim_{z\to 0}\frac{f(z\zeta)}{f(z)} = \zeta} \qquad \text{along } z \in \mathbb{C}^* := \mathbb{C}\setminus\{0\} \qquad \forall\, \zeta \in \mathbb{C}, \tag{10}$$

and, finally, to the property of the limit in (10) to be locally uniform with respect to $\zeta \in \mathbb{C}$.

Following the article [14], the property (10) of a mapping $f : D \to \mathbb{C}$ with $f(0) = 0 \in D$ is called its **asymptotic homogeneity** at 0. In the sequel we sometimes write (10) in the shorter form $f(\zeta z) \sim \zeta f(z)$.

In particular, we obtain from (9) under $|z'| = |z|$ that

$$\boxed{\lim_{r\to 0} \frac{\max\limits_{|z|=r} |f(z)|}{\min\limits_{|z|=r} |f(z)|} \ = \ 1} \tag{11}$$

i.e., that the Lavrent'iev characteristic is equal 1 at the origin. It is natural to say in the case of (11) that the mapping $f$ is **conformal by Lavrent'iev** at 0. As we see, the usual conformality implies the conformality by Belinskij and the latter implies the conformality by Lavrent'iev at the origin meaning geometrically that the infinitesimal circle centered at zero is transformed into an infinitesimal circle also centered at zero.

However, condition (10) much more stronger than condition (11). We obtain from (10) also asymptotic preserving angles

$$\boxed{\lim_{z\to 0} \ \arg \left[\frac{f(z\zeta)}{f(z)}\right] = \arg \zeta} \qquad \forall\, \zeta \in \mathbb{C}^* \tag{12}$$

and asymptotic preserving moduli of infinitesimal rings

$$\boxed{\lim_{z\to 0} \frac{|f(z\,\zeta)|}{|f(z)|} \ = \ |\zeta|} \qquad \forall\, \zeta \in \mathbb{C}^*. \tag{13}$$

The latter two geometric properties characterize asymptotic homogeneity and demonstrate that it is close to the usual conformality.

It should be noted that, despite (13), an asymptotically homogeneous map can send radial lines to infinitely winding spirals, as shown by the example $f(z) = ze^{i\sqrt{-\ln|z|}}$, see [4, p. 41]. Moreover, the above Shabat example shows that the conformality by Belinskij admits infinitely great tensions and pressures at the corresponding points.

It was shown in [34] that a quasiconformal mapping $f : D \to \mathbb{C}$, whose complex characteristic $\mu(z)$ is approximately continuous at a point $z_0 \in D$, is differentiable by Belinskij at the point with $\mu_0 = \mu(z_0)$ and, in particular, is asymptotically homogeneous if $\mu(z_0) = 0$. Recall that $\mu(z)$ is called **approximately continuous at the point** $z_0$ if there is a measurable set $E$ such that $\mu(z) \to \mu(z_0)$ as $z \to z_0$ in $E$ and $z_0$ is a point of density for $E$, i.e.,

$$\lim_{\varepsilon\to 0} \frac{|E \cap D(z_0, \varepsilon)|}{|D(z_0, \varepsilon)|} \ = \ 1,$$

where $D(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}$. Note also that, for functions $\mu$ in $L^\infty$, the points of approximate continuity coincide with the Lebesgue points of $\mu$, i.e., such $z_0$

for which

$$\lim_{r\to 0}\frac{1}{r^2}\int\limits_{|z-z_0|<r}|\mu(z)\ -\ \mu(z_0)|\ dm(z)\ =\ 0,$$

where $dm(z) := dxdy$, $z = x+iy$, stands to the Lebesgue measure (area) in $\mathbb{C}$.

**2. More definitions and preliminary remarks.**

The above results on the asymptotic homogeneity, i.e., on the conformality by Belinskij, were extended to some cases of the Beltrami equations with degeneration.

Further $d\,m(z) := dxdy$, $z = x+iy$, corresponds to the Lebesgue measure (area) in $\mathbb{C}$,

$$dS(z)\ :=\ \frac{d\,m(z)}{(1+|z|^2)^2}\ =\ \frac{dxdy}{(1+|z|^2)^2}\ ,\qquad z := x+iy\ ,$$

is an **element of the spherical area** in the extended complex plane $\overline{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$. Set

$$\mathbb{D}(z_0,r)\ :=\ \{z\in\mathbb{C} : |z-z_0|\ <\ r\}\ ,\qquad \mathbb{D}(r)\ :=\ \mathbb{D}(0,r)\ ,\qquad \mathbb{D}\ :=\ \mathbb{D}(0,1).$$

Here we apply the notation of the mean integral value of a function $\varphi$ over the disks $\mathbb{D}(z_0,r)$

$$\fint_{\mathbb{D}(z_0,r)}\varphi(z)\,dm(z)\ :=\ \frac{1}{|\mathbb{D}(z_0,r)|}\int\limits_{\mathbb{D}(z_0,r)}\varphi(z)\,dm(z).$$

Moreover, the continuity of functions $\Phi : \overline{\mathbb{R}^+}\to\overline{\mathbb{R}^+}$ is understood with respect to the topology of $\overline{\mathbb{R}^+} := [0,\infty]$. A nondecreasing convex function $\Phi : \overline{\mathbb{R}^+}\to\overline{\mathbb{R}^+}$ is called **strictly convex** if $\lim\limits_{t\to\infty}\Phi(t)/t = \infty$, see e.g. III.3.1.1 in [33]. Here we also use the following notions of the inverse function for monotone functions. Namely, for a non-decreasing function $\Phi : \overline{\mathbb{R}^+}\to\overline{\mathbb{R}^+}$, its inverse function $\Phi^{-1} : \overline{\mathbb{R}^+}\to\overline{\mathbb{R}^+}$ can be well-defined by setting

$$\Phi^{-1}(\tau)\ :=\ \inf_{\Phi(t)\geq\tau} t \tag{14}$$

Here inf is equal to $\infty$ if the set of $t\in[0,\infty]$ such that $\Phi(t)\geq\tau$ is empty. Note that the function $\Phi^{-1}$ is non-decreasing, too.

Recall that a mapping $f : D\to\mathbb{C}$ is called **regular at a point** $z_0\in D$, if $f$ has the total differential at this point, and its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 \neq 0$, see e.g. I.1.6 in [24]. In what follows, a homeomorphism $f$ of the class $W^{1,1}_{\mathrm{loc}}$ is called regular, if $J_f(z) > 0$ a.e. Finally, a regular homeomorphism that satisfies a Beltrami equation a.e. in $D$ is called a **regular homeomorphic solution** of the Beltrami equation in $D$. The notion of such a solution was first introduced in [6].

The next statement was first proved as Theorem 6.1 in the article [15] and it extended the corresponding result in [34] mentioned in Introduction from the non-degenerate case to a degenerate case of the Beltrami equations.

**Proposition 1.** *Let $D$ be a domain in $\mathbb{C}$, $0 \in D$, and let $f : D \to \mathbb{C}$ be a regular homeomorphic solution of the Beltrami equation (1) with $f(0) = 0$. Suppose that*

$$\limsup_{r\to 0} \fint_{\mathbb{D}(r)} \Phi\left(K_\mu(z)\right)\, dm(z) \ < \ \infty \tag{15}$$

*for a nondecreasing convex function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ such that, with some $\sigma > \Phi(+0)$,*

$$\int\limits_{\sigma}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \ = \ \infty. \tag{16}$$

*Then the following assertions are equivalent:*
*1) $f$ is conformal by Belinskij at the origin,*
*2) for all $\zeta \in \mathbb{C}$,*

$$\lim_{\substack{\tau\to 0,\\ \tau>0}} \frac{f(\tau\zeta)}{f(\tau)} = \zeta, \tag{17}$$

*3) for all $\delta \in (0, 1]$, $|z'| \le \delta |z|$ and $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$*

$$\lim_{z\to 0} \left\{ \frac{f(z')}{f(z)} - \frac{z'}{z} \right\} = 0, \tag{18}$$

*4) for all $\zeta \in \mathbb{C}$*

$$\lim_{\substack{z\to 0,\\ z\in\mathbb{C}^*}} \frac{f(z\zeta)}{f(z)} = \zeta, \tag{19}$$

*5) the limit in(19) is uniform in the parameter $\zeta$ on each compact subset of $\mathbb{C}$.*

Thus, the conformality by Belinskij still remains equivalent to the property of asymptotic homogeneity under conditions (15) and (16). Hence we recall the connection of condition (16) with other integral conditions, see e.g. Theorem 2.5 in [41].

**Remark 1.** Let $\Phi : [0, \infty] \to [0, \infty]$ be a non-decreasing function and set

$$H(t) \ = \ \log \Phi(t). \tag{20}$$

Then the equality

$$\int\limits_{\Delta}^{\infty} H'(t)\, \frac{dt}{t} = \infty, \tag{21}$$

implies the equality

$$\int\limits_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty\,, \tag{22}$$

and (22) is equivalent to

$$\int\limits_{\Delta}^{\infty} H(t)\,\frac{dt}{t^2} = \infty \tag{23}$$

for some $\Delta > 0$, and (23) is equivalent to each of the equalities

$$\int\limits_{0}^{\delta_*} H\left(\frac{1}{t}\right)\,dt = \infty \tag{24}$$

for some $\delta_* > 0$,

$$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{25}$$

for some $\Delta_* > H(+0)$ and to (16) for some $\delta > \Phi(+0)$.

Moreover, (21) is equivalent to (22) and hence to (23)–(25) as well as to (16) are equivalent to each other if $\Phi$ is in addition absolutely continuous. In particular, all the given conditions are equivalent if $\Phi$ is convex and non-decreasing.

Note that the integral in (22) is understood as the Lebesgue–Stieltjes integral and the integrals in (21) and (23)–(25) as the ordinary Lebesgue integrals. It is necessary to give one more explanation. From the right hand sides in the conditions (21)–(25) we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*]$, then $H(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (22) and (23) exclude that $t_*$ belongs to the interval of integrability because in the contrary case the left hand sides in (22) and (23) are either equal to $-\infty$ or indeterminate. Hence we may assume in (21)–(24) that $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 := \sup\limits_{\Phi(t)=0} t$, and set $t_0 = 0$ if $\Phi(0) > 0$.

The most interesting of the above conditions is (23) that can be rewritten in the form:

$$\int\limits_{\Delta}^{\infty} \log\,\Phi(t)\,\frac{dt}{t^2} \;=\; +\infty \qquad \text{for some } \Delta > 0. \tag{26}$$

Thus, by Theorem 6.2 in [15] we have the next conclusion.

**Proposition 2.** *Let $D$ be a domain in $\mathbb{C}$ and let $f : D \to \mathbb{C}$ be a regular solution of the Beltrami equation* (1)*. Suppose that*

$$\limsup_{r\to 0} \;\;-\!\!\!\!\!\!\int_{\mathbb{D}(z_0,r)} \Phi\left(K_\mu(z)\right)\,dm(z) \;<\; \infty \tag{27}$$

*for a nondecreasing strictly convex function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ with condition* (26)*. If $\mu(z)$ is approximatively continuous at a point $z_0 \in D$, then the mapping $f$ is differentiable by Belinskij at this point with $\mu_0 = \mu(z_0)$.*

**Corollary 1.** *Let $D$ be a domain in $\mathbb{C}$, $0 \in D$, and let $f : D \to \mathbb{C}$ be a regular homeomorphic solution of the Beltrami equation* (1) *with $f(0) = 0$,* (15) *and* (26) *for a nondecreasing strictly convex function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$. Suppose that*

$$\lim_{r\to 0} \frac{1}{r^2} \int\limits_{|z|<r} |\mu(z)| \, dm(z) \ = \ 0. \tag{28}$$

*Then $f$ is asymptotically homogeneous at the origin.*

**Corollary 2.** *Under hypotheses of Corollary 1, $f$ preserves infinitesimal circles centered at the origin, i.e.,*

$$\lim_{r\to 0} \frac{\max\limits_{|z|=r} |f(z)|}{\min\limits_{|z|=r} |f(z)|} \ = \ 1, \tag{29}$$

*asymptotically preserves angles, i.e.,*

$$\lim_{z\to 0} \ \arg \left[\frac{f(z\zeta)}{f(z)}\right] = \arg \zeta \qquad \forall \ \zeta \in \mathbb{C} \ , \ |\zeta| \ = \ 1, \tag{30}$$

*and asymptotically preserves the moduli of infinitesimal rings, i.e.,*

$$\lim_{z\to 0} \frac{|f(z\,\zeta)|}{|f(z)|} \ = \ |\zeta| \qquad \forall \ \zeta \in \mathbb{C}^* \ := \ \mathbb{C} \setminus \{0\}. \tag{31}$$

Moreover, by Theorem 6.3 in [15] we have also the next assertion on logarithms.

**Corollary 3.** *Under hypotheses of Corollary 1,*

$$\lim_{z\to 0} \frac{\ln |f(z)|}{\ln |z|} \ = \ 1. \tag{32}$$

Recall also that a real-valued function $u$ in a domain $D$ in $\mathbb{C}$ is said to be of **bounded mean oscillation** in $D$, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int\limits_B |u(z) - u_B| \, dm(z) < \infty \,, \tag{33}$$

where the supremum is taken over all discs $B$ in $D$ and

$$u_B = \frac{1}{|B|} \int\limits_B u(z) \, dm(z) \,.$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain $U$ of $D$ (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what $D$ is).

The class BMO was introduced by John and Nirenberg (1961) in the paper [22] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see e.g. [19] and [32].

A function $\varphi$ in BMO is said to have **vanishing mean oscillation**, abbr. $\varphi\in$ VMO, if the supremum in (33) taken over all balls $B$ in $D$ with $|B|<\varepsilon$ converges to 0 as $\varepsilon\to 0$. VMO has been introduced by Sarason in [42]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see e.g. [10, 21, 26, 29, 30] and [31].

**Remark 2.** Note that $W^{1,2}(D)\subset VMO(D)$, see e.g. [9].

Following [20], we say that a function $\varphi: D\to\mathbb{R}$ has **finite mean oscillation** at a point $z_0\in D$, abbr. $\varphi\in \mathrm{FMO}(z_0)$, if

$$\overline{\lim_{\varepsilon\to 0}}\ \fint_{B(z_0,\varepsilon)}|\varphi(z)-\widetilde{\varphi}_{\varepsilon}(z_0)|\,dm(z)<\infty\,, \tag{34}$$

where

$$\widetilde{\varphi}_{\varepsilon}(z_0)=\fint_{B(z_0,\varepsilon)}\varphi(z)\,dm(z) \tag{35}$$

is the mean value of the function $\varphi(z)$ over the disk $B(z_0,\varepsilon):=\{z\in\mathbb{C}:|z-z_0|<\varepsilon\}$. Note that the condition (34) includes the assumption that $\varphi$ is integrable in some neighborhood of the point $z_0$. We say also that a function $\varphi: D\to\mathbb{R}$ is of **finite mean oscillation in** $D$, abbr. $\varphi\in\mathrm{FMO}(D)$ or simply $\varphi\in$ FMO, if $\varphi\in\mathrm{FMO}(z_0)$ for all points $z_0\in D$. We write $\varphi\in\mathrm{FMO}(\overline{D})$ if $\varphi$ is given in a domain $G$ in $\mathbb{C}$ such that $\overline{D}\subset G$ and $\varphi\in\mathrm{FMO}(G)$.

The following statement is obvious by the triangle inequality.

**Proposition 3.** *If, for a collection of numbers $\varphi_\varepsilon\in\mathbb{R}$, $\varepsilon\in(0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}}\ \fint_{B(z_0,\varepsilon)}|\varphi(z)-\varphi_{\varepsilon}|\,dm(z)<\infty\,, \tag{36}$$

*then $\varphi$ is of finite mean oscillation at $z_0$.*

In particular, choosing here $\varphi_\varepsilon\equiv 0$, $\varepsilon\in(0,\varepsilon_0]$ in Proposition 3, we obtain the following.

**Corollary 4.** *If, for a point $z_0\in D$,*

$$\overline{\lim_{\varepsilon\to 0}}\ \fint_{B(z_0,\varepsilon)}|\varphi(z)|\,dm(z)<\infty\,, \tag{37}$$

*then $\varphi$ has finite mean oscillation at $z_0$.*

Recall that a point $z_0\in D$ is called a **Lebesgue point** of a function $\varphi: D\to\mathbb{R}$ if $\varphi$ is integrable in a neighborhood of $z_0$ and

$$\lim_{\varepsilon\to 0}\ \fint_{B(z_0,\varepsilon)}|\varphi(z)-\varphi(z_0)|\,dm(z)=0\,. \tag{38}$$

It is known that, almost every point in $D$ is a Lebesgue point for every function $\varphi \in L^1(D)$. Thus, we have by Proposition 3 the next corollary.

**Corollary 5.** *Every locally integrable function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at almost every point in $D$.*

**Remark 3.** Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk $\Delta$, see, e.g., [32, p. 5], and hence also to FMO. However, $\widetilde{\varphi}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (37) is only sufficient but not necessary for a function $\varphi$ to be of finite mean oscillation at $z_0$. Clearly, $\mathrm{BMO}(D) \subset \mathrm{BMO}_{\mathrm{loc}}(D) \subset \mathrm{FMO}(D)$ and as well-known $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$, see, e.g., [22] or [32]. However, FMO is not a subclass of $L^p_{\mathrm{loc}}$ for any $p > 1$ but only of $L^1_{\mathrm{loc}}$. Thus, the class FMO is much more wider than $\mathrm{BMO}_{\mathrm{loc}}$.

Versions of the next lemma has been first proved for the class BMO in [36]. For the FMO case, see the papers [20, 35, 38, 39] and the monographs [16] and [25].

**Lemma 1.** *Let $D$ be a domain in $\mathbb{C}$ and let $\varphi : D \to \mathbb{R}$ be a non-negative function of the class $\mathrm{FMO}(z_0)$ for some $z_0 \in D$. Then*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{\varphi(z)\, dm(z)}{\left(|z-z_0| \log \frac{1}{|z-z_0|}\right)^2} = O\left(\log\log \frac{1}{\varepsilon}\right) \qquad as \quad \varepsilon \to 0 \tag{39}$$

*for some $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 = \min(e^{-e}, d_0)$, $d_0 = \sup\limits_{z\in D} |z - z_0|$.*

The following statement will be also useful later on, see e.g. Theorem 3.2 in [41].

**Proposition 4.** *Let $Q : \mathbb{D} \to [0, \infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{D}} \Phi(Q(z))\, dm(z) < \infty \tag{40}$$

*where $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function such that*

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{41}$$

*for some $\delta > \Phi(+0)$. Then*

$$\int\limits_0^1 \frac{dr}{r q(r)} = \infty \tag{42}$$

*where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.*

**3. On homeomorphic solutions in extended complex plane.**

Here we start from establishing a series of criteria for existence of regular homeomorphic solutions $f : \mathbb{C} \to \mathbb{C}$ to the degenerate Beltrami equations in the whole complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

It is easy to give examples of locally quasiconformal mappings of $\mathbb{C}$ onto the unit disk $\mathbb{D}$, consequently, there exist locally uniform elliptic Beltrami equations with no such solutions. Hence, compared with our previous articles, the main goal here is to find the corresponding additional conditions on dilatation quotients of the Beltrami equations at infinity.

**Lemma 2.** *Let a function $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$. Suppose that, for every $z_0 \in \overline{\mathbb{C}}$, there exist $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$ such that*

$$I_{z_0}(\varepsilon) \ : \ = \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\ dt \ < \ \infty \qquad \forall\, \varepsilon \in (0, \varepsilon_0) \tag{43}$$

*and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\cdot \psi^2_{z_0,\varepsilon}(|z-z_0|)\ dm(z) \ = \ o(I^2_{z_0}(\varepsilon)) \qquad \text{as } \varepsilon \to 0 \quad \forall\, z_0 \in \mathbb{C} \tag{44}$$

*and, moreover,*

$$\int\limits_{\varepsilon<|\zeta|<\varepsilon_\infty} K^T_\mu(\zeta,\infty)\cdot \psi^2_{\infty,\varepsilon}(|\zeta|)\ \frac{dm(\zeta)}{|\zeta|^4} \ = \ o(I^2_\infty(\varepsilon)) \qquad \text{as } \varepsilon \to 0, \tag{45}$$

*where $K^T_\mu(\zeta,\infty) := K^T_\mu(1/\zeta, 0)$.*

*Then the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

**Remark 4.** After the replacements of variables $\zeta \longmapsto z := 1/\zeta$, $\varepsilon \longmapsto R := 1/\varepsilon$, $\varepsilon_\infty \longmapsto R_0 := 1/\varepsilon_\infty$ and functions $\psi_{\infty,\varepsilon}(t) \longmapsto \psi_R(t) := \psi_{\infty,1/R}(1/t)$, the condition (45) can be rewritten in the more convenient form:

$$\int\limits_{R_0<|z|<R} K^T_\mu(z,0)\ \psi^2_R(|z|)\ \frac{dm(z)}{|z|^4} \ = \ o(I^2(R)) \qquad \text{as } R \to \infty, \tag{46}$$

with the family of measurable functions $\psi_R : (0,\infty) \to (0,\infty)$ such that

$$I(R) \ : \ = \int\limits_{R_0}^{R} \psi_R(t)\ \frac{dt}{t^2} \ < \ \infty \qquad \forall\, R \in (R_0, \infty). \tag{47}$$

Before to come to the proof of Lemma 2, let us recall that a **condenser** in $\mathbb{C}$ is a domain $\mathcal{R}$ in $\mathbb{C}$ whose complement in $\overline{\mathbb{C}}$ is the union of two distinguished disjoint compact sets $C_1$ and $C_2$. For convenience, it is written $\mathcal{R} = \mathcal{R}(C_1, C_2)$. A **ring** in $\mathbb{C}$ is a condenser $\mathcal{R} = \mathcal{R}(C_1, C_2)$ with connected $C_1$ and $C_2$ that are called the **complementary components** of $\mathcal{R}$. It is known that the (conformal) capacity of a ring $\mathcal{R} = \mathcal{R}(C_1, C_2)$ in $\mathbb{C}$ is equal to the (conformal) modulus of all paths in $\mathcal{R}$ connecting $C_1$ and $C_2$, see e.g. Theorem A.8 in [25].

*Proof.* By Lemma 3 and Remark 2 in [40] the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ under the conditions (44). Applying conformal (linear) transformation of the plane $\mathbb{C}$, generating by suitable shift, tension and rotation, we may assume that $f(0) = 0$ and $f(1) = 1$. Moreover, by Lemma 3 in [40] we may also assume that $f$ is a ring $Q-$homeomorphism with $Q(z) = K^T_\mu(z, 0)$ at the origin, i.e., for every ring $A = A(r_1, r_2) := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, we have the estimate of the capacity $C_f(r_1, r_2)$ of its image under the mapping $f$:

$$C_f(r_1, r_2) \ \le \int\limits_{A(r_1,r_2)} K^T_\mu(z, 0)\ dm(z) \qquad \forall\, r_1, r_2 :\ 0 < r_1 < r_2 < \infty.$$

Let us consider the mapping $F(z) := 1/f(1/z)$ in $\mathbb{C}_* := \overline{\mathbb{C}} \setminus \{0\}$. Note that $F(\infty) = \infty$ because $f(0) = 0$. Since the capacity is invariant under conformal mappings, we have by the change of variables $z \longmapsto \zeta := 1/z$ as well as $r_1 \longmapsto \varepsilon_2 := 1/r_1$ and $r_2 \longmapsto \varepsilon_1 := 1/r_2$ that

$$C_F(\varepsilon_1, \varepsilon_2) \ \le \int\limits_{A(\varepsilon_1,\varepsilon_2)} K^T_\mu(1/\zeta, 0)\ \frac{dm(\zeta)}{|\zeta|^4} \qquad \forall\, \varepsilon_1, \varepsilon_2 :\ 0 < \varepsilon_1 < \varepsilon_2 < \infty,$$

i.e., $F$ is a ring $\tilde{Q}-$homeomorphism at the origin with $\tilde{Q}(\zeta) := K^T_\mu(1/\zeta, 0)/|\zeta|^4$. Thus, in view of the condition (45), we obtain by Lemma 6.5 in [16] that $F$ has a continuous extension to the origin. Let us assume that $c := \lim\limits_{\zeta \to 0} F(\zeta) \neq 0$.

However, $\overline{\mathbb{C}}$ is homeomorphic to the sphere $\mathbb{S}^2$ by stereographic projection and hence by the Brouwer theorem in $\mathbb{S}^2$ on the invariance of domain the set $C_* := F(\mathbb{C}_*)$ is open in $\overline{\mathbb{C}}$, see e.g. Theorem 4.8.16 in [43]. Consequently, $c \notin C_*$ because $F$ is a homeomorphism. Then the extended mapping $\tilde{F}$ is a homeomorphism of $\overline{\mathbb{C}}$ into $\mathbb{C}_*$ because $f \neq \infty$ in $\mathbb{C}$. Thus, again by the Brouwer theorem, the set $C := \tilde{F}(\overline{\mathbb{C}})$ is open in $\overline{\mathbb{C}}$ and $0 \in \overline{\mathbb{C}} \setminus C \neq \emptyset$. On the other hand, the set $C$ is compact as a continuous image of the compact space $\overline{\mathbb{C}}$. Hence the set $\overline{\mathbb{C}} \setminus C \neq \emptyset$ is also open in $\overline{\mathbb{C}}$. The latter contradicts the connectivity of $\overline{\mathbb{C}}$, see e.g. Proposition I.1.1 in [11].

The obtained contradiction disproves the assumption that $c \neq 0$. Thus, we have proved that $f$ is extended to a homeomorphism of $\overline{\mathbb{C}}$ onto itself with $f(\infty) = \infty$. ☐

Choosing $\psi_{z_0,\varepsilon}(t) \equiv 1/\left(t\ \log\left(1/t\right)\right)$ in Lemma 2, we obtain by Lemma 1 the following.

**Theorem 1.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$ and*

$$\int\limits_{R_0<|z|<R} K_\mu(z)\ \psi^2(|z|)\ \frac{dm(z)}{|z|^4} \ = \ o(I^2(R)) \qquad \text{as } R \to \infty \tag{48}$$

*for some $R_0 > 0$ and a measurable function $\psi : (0,\infty) \to (0,\infty)$ such that*

$$I(R)\ :\ =\ \int\limits_{R_0}^{R} \psi(t)\ \frac{dt}{t^2}\ <\ \infty \qquad \forall\ R \in (R_0,\infty). \tag{49}$$

*Suppose also that $K^T_\mu(z,z_0) \leqslant Q_{z_0}(z)$ a.e. in $U_{z_0}$ for every point $z_0 \in \mathbb{C}$, a neighborhood $U_{z_0}$ of $z_0$ and a function $Q_{z_0} : U_{z_0} \to [0,\infty]$ in the class* $\mathrm{FMO}(z_0)$.

*Then the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

In particular, by Proposition 3 the conclusion of Theorem 1 holds if every point $z_0 \in \mathbb{C}$ is the Lebesgue point of the function $Q_{z_0}$.

By Corollary 4 we obtain the next nice consequence of Theorem 1, too.

**Corollary 6.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$, (48) and*

$$\overline{\lim_{\varepsilon\to 0}}\ \fint_{\mathbb{D}(z_0,\varepsilon)} K^T_\mu(z,z_0)\, dm(z) < \infty \qquad \forall\ z_0 \in \mathbb{C}\,. \tag{50}$$

*Then the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

By (4), we also obtain the following consequences of Theorem 1.

**Corollary 7.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., (48) and $K_\mu$ have a dominant $Q : \mathbb{C} \to [1,\infty)$ in the class* $\mathrm{BMO}_{\mathrm{loc}}$. *Then the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

**Remark 5.** In particular, the conclusion of Corollary 7 holds if $Q \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ because $W^{1,2}_{\mathrm{loc}} \subset \mathrm{VMO}_{\mathrm{loc}}$, see e.g. [9].

**Corollary 8.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., (48) and $K_\mu(z) \leqslant Q(z)$ a.e. in $\mathbb{C}$ with a function $Q$ in the class* $\mathrm{FMO}(\mathbb{C})$. *Then the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

Similarly, choosing $\psi_{z_0,\varepsilon}(t) \equiv 1/t$ in Lemma 2, we come to the next statement.

**Theorem 2.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$, (48) and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\, \frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad \text{as } \varepsilon \to 0 \qquad \forall\ z_0 \in \mathbb{C} \tag{51}$$

*for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

**Remark 6.** Choosing $\psi_{z_0,\varepsilon}(t) \equiv 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$ in Lemma 2, we are able to replace (51) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K_\mu^T(z,z_0)\,dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \tag{52}$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t\log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log \ldots \log 1/t)$.

Now, choosing in Lemma 2 the functional parameter $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := 1/[tk_\mu^T(z_0,t)]$, where $k_\mu^T(z_0,r)$ is the integral mean value of $K_\mu^T(z,z_0)$ over the circle $S(z_0,r) := \{z \in \mathbb{C} : |z-z_0| = r\}$, we obtain one more important conclusion.

**Theorem 3.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$,* (48) *and*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{rk_\mu^T(z_0,r)} = \infty \qquad \forall\ z_0 \in \mathbb{C} \tag{53}$$

*for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

**Corollary 9.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$,* (48) *and*

$$k_\mu^T(z_0,\varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0 \qquad \forall\ z_0 \in \mathbb{C}\ . \tag{54}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

**Remark 7.** In particular, the conclusion of Corollary 10 holds if

$$K_\mu^T(z,z_0) = O\left(\log\frac{1}{|z-z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\ z_0 \in \mathbb{C}\,. \tag{55}$$

Moreover, the condition (54) can be replaced by the whole series of more weak conditions

$$k_\mu^T(z_0,\varepsilon) = O\left(\left[\log\frac{1}{\varepsilon}\cdot\log\log\frac{1}{\varepsilon}\cdot\ldots\cdot\log\ldots\log\frac{1}{\varepsilon}\right]\right) \qquad \forall\ z_0 \in \mathbb{C}. \tag{56}$$

Combining Theorems 3, Proposition 4 and Remark 1, we obtain the following result.

**Theorem 4.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$,* (48) *and*

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K_\mu^T(z,z_0)\right)\,dm(z) < \infty \qquad \forall\ z_0 \in \mathbb{C} \tag{57}$$

*for a neighborhood $U_{z_0}$ of $z_0$ and a convex non-decreasing function $\Phi_{z_0} : [0,\infty] \to [0,\infty]$ with*

$$\int\limits_{\Delta(z_0)}^{\infty} \log\, \Phi_{z_0}(t)\, \frac{dt}{t^2} \;=\; +\infty \tag{58}$$

*for some $\Delta(z_0) > 0$. Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

**Corollary 10.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$,* (48) *and*

$$\int\limits_{U_{z_0}} e^{\alpha(z_0) K_\mu^T(z,z_0)}\, dm(z) < \infty \qquad \forall\ z_0 \in \mathbb{C} \tag{59}$$

*for some $\alpha(z_0) > 0$ and a neighborhood $U_{z_0}$ of the point $z_0$. Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

Since $K_\mu^T(z,z_0) \leqslant K_\mu(z)$ for $z$ and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 4.

**Corollary 11.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e.,* (48) *and*

$$\int\limits_{C} \Phi\left(K_\mu(z)\right)\, dm(z) < \infty \tag{60}$$

*over each compact $C$ in $\mathbb{C}$ for a convex non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$ with*

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \;=\; +\infty \tag{61}$$

*for some $\delta > 0$. Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

**Corollary 12.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e.,* (48) *and, for some $\alpha > 0$, over each compact $C$ in $\mathbb{C}$,*

$$\int\limits_{C} e^{\alpha K_\mu(z)}\, dm(z) \;<\; \infty\ . \tag{62}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.*

**4. On existence of solutions with asymptotic homogeneity at infinity.**

Let us start from the following general lemma on the existence of regular homeomorphic solutions for the Beltrami equations in $\mathbb{C}$ with asymptotic homogeneity at infinity.

**Lemma 3.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a function measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$,*

$$\int\limits_{|z|>R} |\mu(z)|\, dS(z) \;=\; o\left(\frac{1}{R^2}\right) \tag{63}$$

*and, moreover,*

$$\int\limits_{|z|>R} \Phi\left(K_\mu(z)\right) \frac{dm(z)}{|z|^4} \;=\; O\left(\frac{1}{R^2}\right) \tag{64}$$

*for a nondecreasing strictly convex function $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ with*

$$\int\limits_{\Delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^2} \;=\; +\infty \qquad \textit{for some } \Delta > 0. \tag{65}$$

*Suppose also that, for every $z_0 \in \mathbb{C}$, there exist $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0,\infty) \to (0,\infty)$ such that*

$$I_{z_0}(\varepsilon) \;:=\; \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt \;<\; \infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0) \tag{66}$$

*and*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z,z_0)\cdot\psi^2_{z_0,\varepsilon}(|z-z_0|)\, dm(z) \;=\; o(I^2_{z_0}(\varepsilon)) \qquad \textit{as } \varepsilon \to 0 \quad \forall\, z_0 \in \mathbb{C}\,. \tag{67}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity $f(\zeta z) \sim \zeta f(z)$ as $z \to \infty$ for all $\zeta \in \mathbb{C}$, i.e.,*

$$\boxed{\lim_{\substack{z\to\infty,\\ z\in\mathbb{C}}} \frac{f(z\zeta)}{f(z)} = \zeta} \qquad \forall\, \zeta \in \mathbb{C} \tag{68}$$

*and the limit* (68) *is locally uniform with respect to the parameter $\zeta$ in $\mathbb{C}$.*

**Remark 8.** In particular, the property of the asymptotic homogeneity of $f$ at infinity implies its **conformality by Belinskij at infinity**, i.e.,

$$\boxed{f(z) \;=\; A(\rho)\cdot[\, z \;+\; o(\rho)\,]} \qquad \text{as } z \to \infty, \tag{69}$$

where $A(\rho)$ depends only on $\rho = |z|$, $o(\rho)/\rho \to 0$ as $\rho \to \infty$ and, moreover,

$$\boxed{\lim_{\rho\to\infty} \frac{A(t\rho)}{A(\rho)} = 1} \qquad \forall\, t > 0, \tag{70}$$

its **conformality by Lavrent'iev at infinity**, i.e.,

$$\boxed{\lim_{R\to\infty} \frac{\max\limits_{|z|=R} |f(z)|}{\min\limits_{|z|=R} |f(z)|} \ = \ 1,} \tag{71}$$

the logarithmic property at infinity

$$\lim_{z\to\infty} \frac{\ln |f(z)|}{\ln |z|} \ = \ 1, \tag{72}$$

asymptotic preserving angles at infinity, i.e.,

$$\boxed{\lim_{z\to\infty} \arg \left[\frac{f(z\zeta)}{f(z)}\right] = \arg\zeta} \qquad \forall\, \zeta \in \mathbb{C}^* \tag{73}$$

and asymptotic preserving moduli of rings at infinity, i.e.,

$$\boxed{\lim_{z\to\infty} \frac{|f(z\,\zeta)|}{|f(z)|} \ = \ |\zeta|} \qquad \forall\, \zeta \in \mathbb{C}^*. \tag{74}$$

The latter two geometric properties characterize asymptotic homogeneity at infinity and demonstrate that it is close to the usual conformality at infinity.

*Proof.* After the replacements of variables $z \longmapsto \xi := 1/z$ and $R \longmapsto r := 1/R$, the condition (64) can be rewritten in the form

$$\limsup_{r\to 0} \ \frac{1}{r^2} \int\limits_{|\xi|<r} \Phi\left(K_\mu\left(\frac{1}{\xi}\right)\right) \ dm(\xi) \ < \ \infty, \tag{75}$$

and the latter implies, in particular, that, for some $r_0 \in (0, 1]$,

$$\frac{1}{r_0^2} \int\limits_{|\xi|<r_0} \Phi\left(K_\mu\left(\frac{1}{\xi}\right)\right) \ dm(\xi) \ < \ \infty, \tag{76}$$

i.e., for $Q(\zeta) := K_\mu(1/r_0\,\zeta)$ and $\zeta := \xi/r_0$, we have that

$$\int\limits_{\mathbb{D}} \Phi(Q(\zeta)) \ dm(\zeta) \ < \ \infty. \tag{77}$$

Consequently, by Proposition 4, see also Remark 1, we obtain that

$$\int\limits_0^1 \frac{d\rho}{\rho\, q(\rho)} \ = \ \infty, \tag{78}$$

where $q(\rho) \geq 1$ is the average of the function $Q$ over the circle $|\zeta| = \rho \in (0,1)$. Hence it is easy to see that

$$\int\limits_{\varepsilon<|\zeta|<1} Q(\zeta) \cdot \psi^2(|\zeta|)\, dm(\zeta) \;=\; 2\pi \cdot I(\varepsilon) \;=\; o(I^2(\varepsilon)) \qquad \varepsilon \to 0, \tag{79}$$

where

$$\psi(t) \;:=\; \frac{1}{tq(t)}\,, \qquad I(\varepsilon) \;:=\; \int\limits_{\varepsilon}^{1} \psi(t)dt \;<\; \infty \qquad \forall\, \varepsilon \in (0,1). \tag{80}$$

Setting $R_0 := 1/r_0$ and $\psi_*(t) = \psi(1/t)$, we also see from here that

$$\int\limits_{R_0<|z|<R} K_\mu(z)\ \psi_*^2(|z|)\ \frac{dm(z)}{|z|^4} \;=\; o(I_*^2(R)) \qquad as\ R \to \infty, \tag{81}$$

where

$$I_*(R) \;:=\; \int\limits_{R_0}^{R} \psi_*(t)\ \frac{dt}{t^2} \;<\; \infty \qquad \forall\ R \in (R_0, \infty). \tag{82}$$

By (4) and (81)-(82) we obtain also that

$$\int\limits_{R_0<|z|<R} K_\mu^T\,(z,0)\ \ \psi_*^2(|z|)\ \frac{dm(z)}{|z|^4} \;=\; o(I_*^2(R)). \tag{83}$$

Thus, by Lemma 2 and Remark 4, the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Setting $f^*(\xi) := 1/f(1/\xi)$ in $\overline{\mathbb{C}}$, we see that $f^*(0) = 0$, $f^*(1) = 1$, $f^*(\infty) = \infty$ and that $f^*$ is a regular homeomorphic solution in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of the Beltrami equation with

$$\mu^*(\xi) \;:=\; \mu\left(\frac{1}{\xi}\right) \cdot \frac{\xi^2}{\bar{\xi}^2}\,, \qquad K_{\mu^*}(\xi) \;=\; K_\mu\left(\frac{1}{\xi}\right), \tag{84}$$

because

$$f^*_{\bar{\xi}}(\xi) \;=\; \frac{1}{\bar{\xi}^2} \cdot \frac{f_{\bar{z}}(\frac{1}{\xi})}{f^2(\frac{1}{\xi})}\,, \qquad f^*_{\xi}(\xi) \;=\; \frac{1}{\xi^2} \cdot \frac{f_z(\frac{1}{\xi})}{f^2(\frac{1}{\xi})} \qquad \text{a.e. in } \mathbb{C}, \tag{85}$$

see e.g. Section I.C and the proof of Theorem 3 of Section V.B in [1].

Note that $f^*$ belongs to the class $W^{1,1}_{\rm loc}(\mathbb{C}^*)$ and, consequently, $f^*$ is ACL (absolutely continuous on lines) in $\mathbb{C}$, see e.g. Theorems 1 and 2 of Section 1.1.3 and Theorem of Section 1.1.7 in [27]. However, it is not clear directly from (85) whether the derivatives $f^*_{\bar{\xi}}$ and $f^*_{\xi}$ are integrable in a neighborhood of the origin, because of the first factors in (85). Thus, to prove that $f^*$ is a regular homeomorphic solution of the Beltrami equation in $\mathbb{C}$, it remains to establish the latter fact in another way.

Namely, by (76) and (84), as well as by spherical symmetry of the domains of integrability, we have that

$$\int\limits_{|\xi|<r_0} \Phi(K_{\mu^*}(\xi))\ dm(\xi)\ <\ \infty \tag{86}$$

for some $r_0 \in (0,1]$ because the function $\Phi$ is nondecreasing. Moreover, since $\Phi$ is strictly convex, in particular, $\Phi(t)/t \to 0$ as $t \to \infty$, we obtain that there is $t_0 > 1$ such that $\Phi(t)/t > 1$ for all $t > t_0$. Thus, (86) implies that

$$\int\limits_{|\xi|<r_0} K_{\mu^*}(\xi)\ dm(\xi)\ \le\ \pi t_0 r_0^2\ +\ \int\limits_{|\xi|<r_0} \Phi(K_{\mu^*}(\xi))\ dm(\xi)\ <\ \infty, \tag{87}$$

i.e., the dilatation quotient $K_{\mu^*}$ of the given Beltrami equation is integrable in the disk $\mathbb{D}(r_0)$.

Now, since $f^*$ is a regular homeomorphism in $\mathbb{C}^*$, in particular, its Jacobian $J(\xi) = |f^*_\xi|^2 - |f^*_{\bar\xi}|^2 \neq 0$ a.e. and hence $|f^*_\xi| - |f^*_{\bar\xi}| \neq 0$ a.e. as well as $f^*_\xi \neq 0$ a.e., the following identities are also correct a.e.

$$|f^*_\xi(\xi)|\ +\ |f^*_{\bar\xi}(\xi)|\ =\ \left[\frac{|f^*_\xi(\xi)|\ +\ |f^*_{\bar\xi}(\xi)|}{|f^*_\xi(\xi)|\ -\ |f^*_{\bar\xi}(\xi)|}\right]^{\frac{1}{2}} \cdot\ J^{\frac{1}{2}}(\xi)\ =\ K^{\frac{1}{2}}_{\mu^*}(\xi)\cdot J^{\frac{1}{2}}(\xi). \tag{88}$$

Hence by the Hölder inequality for integrals, see e.g. Theorem 189 in [18], we have that

$$\int\limits_{|\xi|<r_0} \left(|f^*_\xi(\xi)|\ +\ |f^*_{\bar\xi}(\xi)|\right)\ dm(\xi)\ \le\ \left(\int\limits_{|\xi|<r_0} K_{\mu^*}(\xi)\, dm(\xi)\right)^{\frac{1}{2}} \cdot \left(\int\limits_{|\xi|<r_0} J(\xi)\, dm(\xi)\right)^{\frac{1}{2}} \tag{89}$$

and, since the latter factor in (89) is estimated by the area of $f^*(\mathbb{D}(r_0)$, see e.g. the Lebesgue theorem in Section III.2.3 of the monograph [24], we conclude that both partial derivatives $f^*_\xi$ and $f^*_{\bar\xi}$ are integrable in the disk $\mathbb{D}(r_0)$.

Next, since the function $\Phi$ is nondecreasing, we have by (75) and (84), as well as by spherical symmetry of the domains of integrability, that

$$\limsup_{r\to 0}\ \frac{1}{r^2} \int\limits_{|\xi|<r} \Phi\left(K_{\mu^*}(\xi)\right)\ dm(\xi)\ <\ \infty. \tag{90}$$

Moreover, by (63) and (84), we obtain that

$$\lim_{r\to 0}\ \frac{1}{r^2} \int\limits_{|\xi|<r} |\mu^*(z)|\ dm(z)\ =\ 0. \tag{91}$$

Thus, by Theorems 6.1 and 6.2 in [15], see also Remark 1, we conclude that $f^*$ is asymptotically homogeneous at the origin, i.e.,

$$\lim_{\substack{\xi\to 0,\\ \xi\in\mathbb{C}^*}} \frac{f^*(\xi\zeta)}{f^*(\xi)} = \zeta \qquad \forall\, \zeta\in\mathbb{C} \tag{92}$$

and, furthermore, the limit in(92) is locally uniform in the parameter $\zeta$.

After the replacements of variables $\xi \longmapsto w{:=}1/\xi$ and $f^*(\xi) \longmapsto f(w){=}1/f^*(1/w)$ the relation (92) can be rewritten in the form

$$\lim_{\substack{w\to\infty,\\ w\in\mathbb{C}}} \frac{f(w)}{f\left(w\zeta^{-1}\right)} = \zeta \qquad \forall\, \zeta\in\mathbb{C}\ . \tag{93}$$

Finally, after the change of variables $w \longmapsto z := w\zeta^{-1}$, the latter is transformed into (68), where the limit is locally uniform with respect to the parameter $\zeta\in\mathbb{C}$. $\Box$

Choosing $\psi_{z_0,\varepsilon}(t)\equiv 1/\left(t\,\log\left(1/t\right)\right)$ in Lemma 3, we obtain by Lemma 1 the following.

**Theorem 5.** *Let $\mu:\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e., $K_\mu\in L^1_{\rm loc}(\mathbb{C})$,* (63)*–*(65)*. Suppose also that $K^T_\mu(z,z_0)\leqslant Q_{z_0}(z)$ a.e. in $U_{z_0}$ for every point $z_0\in\mathbb{C}$, a neighborhood $U_{z_0}$ of $z_0$ and a function $Q_{z_0}:U_{z_0}\to[0,\infty]$ in the class* FMO$(z_0)$*. Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0)=0$, $f(1)=1$ and $f(\infty)=\infty$ that is asymptotically homogeneous at infinity.*

In particular, by Proposition 3 the conclusion of Theorem 5 holds if every point $z_0\in\mathbb{C}$ is the Lebesgue point of the function $Q_{z_0}$.

By Corollary 4 we obtain the next fine consequence of Theorem 5, too.

**Corollary 13.** *Let $\mu:\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e., $K_\mu\in L^1_{\rm loc}(\mathbb{C})$,* (63)*–*(65) *and*

$$\overline{\lim_{\varepsilon\to 0}}\ \ \fint_{\mathbb{D}(z_0,\varepsilon)} K^T_\mu(z,z_0)\,dm(z)<\infty \qquad \forall\ z_0\in\mathbb{C}\,. \tag{94}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0)=0$, $f(1)=1$ and $f(\infty)=\infty$ that is asymptotically homogeneous at infinity.*

By (4), we also obtain the following consequences of Theorem 5.

**Corollary 14.** *Let $\mu:\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e.,* (63)*–*(65) *and $K_\mu$ have a dominant $Q:\mathbb{C}\to[1,\infty)$ in the class* BMO$_{\rm loc}$*. Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0)=0$, $f(1)=1$ and $f(\infty)=\infty$ that is asymptotically homogeneous at infinity.*

**Remark 9.** In particular, the conclusion of Corollary 14 holds if $Q\in \mathrm{W}^{1,2}_{\rm loc}$ because $W^{1,2}_{\rm loc}\subset \mathrm{VMO}_{\rm loc}$, see e.g. [9].

**Corollary 15.** *Let $\mu:\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e.,* (63)*–*(65) *and $K_\mu(z)\leqslant Q(z)$ a.e. in $\mathbb{C}$ with a function $Q$ in the class* FMO$(\mathbb{C})$*. Then* *the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.*

Similarly, choosing $\psi_{z_0,\varepsilon}(t) \equiv 1/t$ in Lemma 3, we come to the next statement.

**Theorem 6.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$,* (63)–(65) *and, for some $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K^T_\mu(z, z_0)\,\frac{dm(z)}{|z-z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad \textit{as } \varepsilon \to 0 \qquad \forall\ z_0 \in \mathbb{C}. \tag{95}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.*

**Remark 10.** Choosing $\psi_{z_0,\varepsilon}(t) \equiv 1/(t\log 1/t)$ instead of $\psi(t) = 1/t$ in Lemma 3, we are able to replace (51) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K^T_\mu(z, z_0)\,dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \qquad \text{as } \varepsilon \to 0 \qquad \forall\ z_0 \in \mathbb{C}. \tag{96}$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t\log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log\ldots\log 1/t)$.

Now, choosing in Lemma 3 the functional parameter $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t) := 1/[tk^T_\mu(z_0, t)]$, where $k^T_\mu(z_0, r)$ is the integral mean value of $K^T_\mu(z, z_0)$ over the circle $S(z_0, r) := \{z \in \mathbb{C} : |z-z_0| = r\}$, we obtain one more important conclusion.

**Theorem 7.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$,* (63)–(65) *and, for some $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{rk^T_\mu(z_0, r)} = \infty \qquad \forall\ z_0 \in \mathbb{C}. \tag{97}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.*

**Corollary 16.** *Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\mathrm{loc}}(\mathbb{C})$,* (63)–(65) *and*

$$k^T_\mu(z_0, \varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \qquad \textit{as } \varepsilon \to 0 \qquad \forall\ z_0 \in \mathbb{C}. \tag{98}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.*

**Remark 11.** In particular, the conclusion of Corollary 10 holds if

$$K^T_\mu(z,z_0) = O\left(\log\frac{1}{|z-z_0|}\right) \qquad \text{as} \quad z\to z_0 \quad \forall\, z_0\in\mathbb{C}\,. \tag{99}$$

Moreover, the condition (54) can be replaced by the whole series of more weak conditions

$$k^T_\mu(z_0,\varepsilon) = O\left(\left[\log\frac{1}{\varepsilon}\cdot\log\log\frac{1}{\varepsilon}\cdot\ldots\cdot\log\ldots\log\frac{1}{\varepsilon}\right]\right) \qquad \forall\, z_0\in\mathbb{C}. \tag{100}$$

Combining Theorems 7, Proposition 4 and Remark 1, we obtain the following result.

**Theorem 8.** *Let $\mu:\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e., $K_\mu\in L^1_{\rm loc}(\mathbb{C})$, (63)–(65) and*

$$\int\limits_{U_{z_0}} \Phi_{z_0}\left(K^T_\mu(z,z_0)\right)\, dm(z) < \infty \qquad \forall\, z_0\in\mathbb{C} \tag{101}$$

*for a neighborhood $U_{z_0}$ of $z_0$ and a convex non-decreasing function $\Phi_{z_0}:[0,\infty]\to[0,\infty]$ with*

$$\int\limits_{\Delta(z_0)}^{\infty} \log\,\Phi_{z_0}(t)\,\frac{dt}{t^2} \;=\; +\infty \qquad \text{for some } \Delta(z_0)>0. \tag{102}$$

*Then the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0)=0$, $f(1)=1$ and $f(\infty)=\infty$ that is asymptotically homogeneous at infinity.*

**Corollary 17.** *Let $\mu:\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e., (63)–(65) and, for some $\alpha(z_0)>0$ and a neighborhood $U_{z_0}$ of the point $z_0$,*

$$\int\limits_{U_{z_0}} e^{\alpha(z_0)K^T_\mu(z,z_0)}\, dm(z) < \infty \qquad \forall\, z_0\in\mathbb{C}. \tag{103}$$

*Then the Beltrami equation (1) has a regular homeomorphic solution $f$ in $\mathbb{C}$ with $f(0)=0$, $f(1)=1$ and $f(\infty)=\infty$ that is asymptotically homogeneous at infinity.*

Since $K^T_\mu(z,z_0)\leqslant K_\mu(z)$ for $z$ and $z_0\in\mathbb{C}$, we also obtain the following consequences of Theorem 8.

**Corollary 18.** *Let $\mu:\mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e., (63)–(65) and over each compact $C$ in $\mathbb{C}$*

$$\int\limits_{C} \Phi\left(K_\mu(z)\right)\, dm(z) < \infty \tag{104}$$

*for a convex non-decreasing function* $\Phi : [0,\infty] \to [0,\infty]$ *such that, for some* $\delta > 0$,

$$\int\limits_{\delta}^{\infty} \log\,\Phi(t)\,\frac{dt}{t^2} \;=\; +\infty. \tag{105}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution* $f$ *in* $\mathbb{C}$ *with* $f(0) = 0$, $f(1) = 1$ *and* $f(\infty) = \infty$ *that is asymptotically homogeneous at infinity.*

**Corollary 19.** *Let* $\mu : \mathbb{C} \to \mathbb{C}$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* (63)–(65) *and, for some* $\alpha > 0$, *over each compact* $C$ *in* $\mathbb{C}$

$$\int\limits_{C} e^{\alpha K_\mu(z)}\,dm(z) \;<\; \infty. \tag{106}$$

*Then the Beltrami equation* (1) *has a regular homeomorphic solution* $f$ *in* $\mathbb{C}$ *with* $f(0) = 0$, $f(1) = 1$ *and* $f(\infty) = \infty$ *that is asymptotically homogeneous at infinity.*

**Remark 12.** Recall that by Theorem 5.1 in [41] the condition (105) is not only sufficient but also necessary for the existence of regular homeomorphic solutions for all Beltrami equations (1) with the integral constraints (104), see also Remark 1.

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**В. Гутлянський, В. Рязанов, Є. Севостьянов, Е. Якубов**

**Про відображення з асимптотичною однорідністю на нескінченності і рівняння Бельтрамі.**

У цій статті встановлено критерії для існування регулярних гомеоморфних розв'язкив вироджених рівнянь Бельтрамі у комплексній площині $\mathbb{C}$ з нормалізацією $f(0) = 0$, $f(1) = 1$, $f(\infty) = \infty$ та з асимптотичною однорідністю у нескінченності. Остання властивість тягне, зокрема, конформність за Білинським, а також конформність за Лаврентьєвим на нескінченності. Вступ містить необхідну інформацію про історію питання, яке повертається до монографії відомого спеціаліста в теорії квазіконформних відображень, професора Павла Петровича Білинського. Автори присвячують свою статтю його пам'яті. Перші два авторі статті були свого часу безпосередньо пов'язані з подальшим розвитком цієї теми і, зокрема, встановили еквівалентність конформністі за Білинським асимптотичної однорідності у скінчених точках площини для квазіконформних відображень та їх узагальнень. Другий розділ містить відповідні означення та формулювання у випадку вироджених рівнянь Бельтрамі. Крім того, цей розділ містить необхідну інформацію про еквівалентність цілого ряду інтегральних умов, а також критерії для скінченого середнього коливання функцій у точці, які використовуються в наступних розділах для доказу основних результатів. У третьому розділі роботи автори встановили ряд критеріїв для існування регулярних гомеоморфних розв'язків вироджених рівнянь Бельтрамі у комплексній площині з нормалізацією $f(0) = 0$, $f(1) = 1$ і $f(\infty) = \infty$. Починається розділ із загальної леми 2, яка формулюється в термінах сингулярних функціональних параметрів. На цій основі, потім виводиться багато інших критеріїв. У цьому розділі інтегральні умови на нескінченності залишаються з сингулярними функціональними параметрами загальної форми, а у скінчених точках площини мають особливий вигляд у різних критеріях. Останній четвертий розділ також починається із загальної леми 3 та присвячено існуванню регулярних гомеоморфних розв'язків вироджених рівнянь Бельтрамі у комплексній площині з асимптотичною однорідністю на нескінченності. Тут умова на нескінченності посилюється, а умови у скінчених точках залишаються незмінними. Ми згадуємо тут лише найцікавіші з них. Перш за все, це є критерій в наслідку 14 існування домінанти для дилатаційного відношення рівняння у відомому класу BMO (функцій з обмеженим середнім коливанням) по Джону–Ніренбергу, та інший більш загальний критерій в наслідку 15 існування його домінанти в класі FMO (функції з скінченим середнім коливанням). Іншими словами, останнє означає що така домінанта має скінченну дисперсію над нескінченно малими дисками. Зокрема, це так, якщо середнє значення дилатації над нескінченно малими дисками скінчено, див. наслідок 13. Далі, теорема 6 та зауваження 10 містять критерії в термінах контрольованої розбіжності сингулярних інтегралів типу Кальдерона–Зигмунда. З леми 3 також випливає теорема 7 з критерієм типу Лаврентьєва–Лехто про середні значення уздовж кіл та наслідок 16 і зауваження 11, які показують, що висновок має місце, якщо дилатація зростає не швидше, ніж логарифм відстані до

центрів кіл. Нарешті, наслідок у теоремі 8 дає інтегральний критерій типу Орліча та наслідок 19 говорить про прості інтегральні умови експоненціального типу.

***Ключові слова:*** *вироджені рівняння Бельтрамі, конформність за Білинським та за Лаврентьєвим, асимптотична однорідність у нескінченності.*

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*Received 01.12.21*

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