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МІЖНАРОДНА НАУКОВА КОНФЕРЕНЦІЯ

«СУЧАСНІ ПРОБЛЕМИ МАТЕМАТИЧНОГО АНАЛІЗУ І ТЕОРІЇ ФУНКЦІЙ»

ТЕЗИ ДОПОВІДЕЙ

ЖИТОМИР, УКРАЇНА

24 – 25 ВЕРЕСНЯ 2026

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МІЖНАРОДНА НАУКОВА КОНФЕРЕНЦІЯ "СУЧАСНІ ПРОБЛЕМИ МАТЕМАТИЧНОГО АНАЛІЗУ І ТЕОРІЇ ФУНКЦІЙ"

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С91 Сучасні проблеми математичного аналізу і теорії функцій: збірник тез доповідей Міжнародної наукової конференції, 24–25 вересня / за заг. редакцією проф. Є. О. Севостьянова. — Житомир, 2026. — 107 с.

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Збірник розрахований на науковців, викладачів, аспірантів, студентів та всіх, хто цікавиться сучасними проблемами математичного аналізу і теорії функцій.

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INTERNATIONAL SCIENTIFIC CONFERENCE "MODERN PROBLEMS OF MATHEMATICAL ANALYSIS AND FUNCTION THEORY"

UKRAINE, ZHYTOMYR
SEPTEMBER 24 — 25, 2026

ABSTRACTS

Zhytomyr 2026

International Scientific Conference "Modern Problems of Mathematical Analysis and Function Theory", Zhytomyr Ivan Franko State University

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Sections of the conference:

1. Hypercomplex Analysis

2. Extremal Problems

3. Problems of Mathematical Physics

4. Mathematical and Complex Analysis

5. Probability Theory and Mathematical Statistics

Contents

Секція 1
Гіперкомплексний аналіз / Hypercomplex Analysis

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A HYPERCOMPLEX METHOD OF SOLVING THE DIRICHLET PROBLEM FOR THE LAMÉ SYSTEM

A hypercomplex method for solving the Dirichlet problem for the Lamé system in a bounded simply connected domain is developed. It is based on representing solutions via components of a monogenic function in a commutative biharmonic algebra (they have small differences with respect to similar representations obtained in [1]).

A theory of monogenic functions with values in the biharmonic algebra is well developed by the authors (cf., e.g., [2]).

The Dirichlet problem is reduced to a system of integral equations by using a hypercomplex Cauchy type integral. Sufficient conditions for the Fredholm property of the specified system are established for bounded domains with a smooth boundary. A boundary of such domains belongs to a class which is essentially wider than the class of Lyapunov curves. At the same time we assume that given boundary functions belong to certain classes of continuous functions. These classes are wider than the Hölder classes of continuous functions. The solution of the Dirichlet problem for a disk is obtained in explicit form.

All results with full proofs can be found in the paper [3].

R E F E R E N C E S

1. Gryshchuk, S.V. *Representations of solutions of Lamé system with real coefficients via monogenic functions in the biharmonic algebra.* Proc. Intern. Geom. Center., **16** (1), 78 – 90 (2023).

2. Gryshchuk, S.V., Plaksa, S.A. *Basic Properties of Monogenic Functions in a Biharmonic Plane.* In: M. L. Agranovskii, M. Ben-Artzi, G. Galloway, L. Karp, V. Maz'ya (eds.), “Complex Analysis and Dynamical Systems V”. Contemporary Mathematics, **591**, Amer. Math. Soc., Providence, RI, 127–134 (2013).

3. Gryshchuk, S.V., Plaksa, S.A. *Solving the Dirichlet problem for the Lamé system by the hypercomplex method*, Ukr. Mat. Z. **78**(5-6), 339–358 (2026) [Ukrainian].

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ORDERING OF A COMPLEX-VALUED SAMPLE BY THE METHOD OF MUTUAL PLACEMENT OF COMPLEX POINTS

Set theory studies the concept of orderings, according to which every set can be ordered – this is equivalent to the axiom of choice. However, no ordering of the set of complex numbers can satisfy the trichotomy law. There are various plausible ways to order the set of complex numbers, along with explanations for why they fail [1].

This paper analyzes a method for ordering a sample (subset) of complex numbers with respect to the position of complex points on the complex plane and formulates an analogue of the trichotomy law for this approach. The results of this work are published in the article [2].

Let's consider a complex-valued sample (subset) $\mathbb{S} \subset \mathbb{C}$:

$$\mathbb{S} = \{\omega_k = a_k + b_k i, \quad a_k, b_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n\}.$$

Let $\langle \lhd, \rhd \rangle$ denote an ordered pair of comparison symbols $(<, =, >)$ for the real $\mathrm{Re}\,\omega_k$ and imaginary $\mathrm{Im}\,\omega_k$ parts of the complex numbers ω_k.

In that case, the two complex points $\omega_1 = a_1 + b_1 i$ and $\omega_2 = a_2 + b_2 i$ have nine relative positions with respect to each other, corresponding to nine situations:

1) Same place (center):

$$(\omega_1 = \omega_2) := \omega_1 \langle =, = \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 = Re\ \omega_2, \\ Im\ \omega_1 = Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 = a_2, \\ b_1 = b_2. \end{cases}$$

2) Straight up (above):

$$(\omega_1 \preccurlyeq \omega_2) := \omega_1 \langle =, < \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 = Re\ \omega_2, \\ Im\ \omega_1 < Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 = a_2, \\ b_1 < b_2. \end{cases}$$

3) Straight down (below):

$$(\omega_1 \succcurlyeq \omega_2) := \omega_1 \langle =, > \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 = Re\ \omega_2, \\ Im\ \omega_1 > Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 = a_2, \\ b_1 > b_2. \end{cases}$$

4) Straight right (right):

$$(\omega_1 \curlyeqprec \omega_2) := \omega_1 \langle <, = \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 < Re\ \omega_2, \\ Im\ \omega_1 = Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 < a_2, \\ b_1 = b_2. \end{cases}$$

5) Straight left (left):

$$(\omega_1 \curlyeqsucc \omega_2) := \omega_1 \langle >, = \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 > Re\ \omega_2, \\ Im\ \omega_1 = Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 > a_2, \\ b_1 = b_2. \end{cases}$$

6) Right and up (right-above):

$$(\omega_1 \prec \omega_2) := \omega_1 \langle <, < \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 < Re\ \omega_2, \\ Im\ \omega_1 < Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 < a_2, \\ b_1 < b_2. \end{cases}$$

7) Left and down (left-below):

$$(\omega_1 \succ \omega_2) := \omega_1 \langle >, > \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 > Re\ \omega_2, \\ Im\ \omega_1 > Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 > a_2, \\ b_1 > b_2. \end{cases}$$

8) Right and down (right-below):

$$(\omega_1 \prec\!\succ \omega_2) := \omega_1 \langle <, > \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 < Re\ \omega_2, \\ Im\ \omega_1 > Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 < a_2, \\ b_1 > b_2. \end{cases}$$

9) Left and up (left-above):

$$(\omega_1 \succ\!\prec \omega_2) := \omega_1 \langle >, < \rangle \omega_2 \Leftrightarrow \begin{cases} Re\ \omega_1 > Re\ \omega_2, \\ Im\ \omega_1 < Im\ \omega_2. \end{cases} \Leftrightarrow \begin{cases} a_1 > a_2, \\ b_1 < b_2. \end{cases}$$

Lemma [2]. *The complex plane $\mathbb{C}$ can be represented as the union of disjoint subsets $\mathbb{S}_j$:*

$$\mathbb{C} = \bigcup_{j=1}^{9} \mathbb{S}_j, \qquad \mathbb{S}_j \cap \mathbb{S}_l = \varnothing,\ j \neq l, \quad l = 1, \ldots, 9, \tag{1}$$

where

$$\mathbb{S}_1 = \{\omega, w \in \mathbb{C} \mid \omega = w\}\,;\ \mathbb{S}_2 = \{\omega, w \in \mathbb{C} \mid \omega \preccurlyeq w\}\,;\ \mathbb{S}_3 = \{\omega, w \in \mathbb{C} \mid \omega \succcurlyeq w\}\,;$$

$$\mathbb{S}_4 = \{\omega, w \in \mathbb{C} \mid \omega \preceq w\}\,;\ \mathbb{S}_5 = \{\omega, w \in \mathbb{C} \mid \omega \succeq w\}\,;\ \mathbb{S}_6 = \{\omega, w \in \mathbb{C} \mid \omega \prec w\}\,;$$

$$\mathbb{S}_7 = \{\omega, w \in \mathbb{C} \mid \omega \succ w\}\,;\ \mathbb{S}_8 = \{\omega, w \in \mathbb{C} \mid \omega \prec\!\succ w\}\,;\ \mathbb{S}_9 = \{\omega, w \in \mathbb{C} \mid \omega \succ\!\prec w\}\,.$$

We formulate an analogue of the trichotomy law for complex number: *Any two complex numbers can be compared to each other only under one of the nine relations that make up the partition of the complex planes $\mathbb{S}_j$ (1).*

Remark. Although an analogue of the trichotomy law holds for the partition (1) of the complex plane C, this does not imply a complete ordering of the complex plane, but only a partial ordering.

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R E F E R E N C E S

1. Fenwick Ch. *On complex numbers I – Part 1: Introducing complex numbers, graphing complex numbers, doing arithmetic on complex numbers, and the polar form of a complex numbers..* Center for languages and International Education, Institute of Education, University College London, 149 p. (2022).

2. Kolomiiets T. Yu., Kolomiiets R. O., Targonskii A. L. Organizing a complex sample. Proceedings of Institute of Applied Mathematics and Mechanics of NAS of Ukraine. 40(1), 35–49 (2026).

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DIFFERENT CLASSES OF MONOGENIC FUNCTIONS IN THE THREE-DIMENSIONAL NONCOMMUTATIVE ALGEBRA $\widetilde{\mathbb{A}}_2$

Let us consider a three-dimensional noncommutative algebra $\widetilde{\mathbb{A}}_2$ over the field $\mathbb{C}$ with the basis $\{I_1, I_2, \rho\}$, satisfying the following multiplication rules:

$\cdot$	I_1	I_2	ρ
I_1	I_1	0	0
I_2	0	I_2	ρ
ρ	ρ	0	0

,

where unit of the algebra is decomposed as: $1 = I_1 + I_2$.

We consider the triple of the vectors

$$i_1 = a_1 I_1 + a_2 I_2\,, \qquad i_2 = b_1 I_1 + b_2 I_2\,, \qquad i_3 = c_1 I_1 + c_2 I_2\,,$$

where $a_k, b_k, c_k \in \mathbb{C},\ k = 1, 2$. Let us consider the linear span

$$E_3 := \{\zeta = x i_1 + y i_2 + z i_3 : x, y, z \in \mathbb{R}\}$$

over the field $\mathbb{R}$, generated by the vectors i_1, i_2, i_3. The element $\zeta \in E_3$ is represented as $\zeta = \xi_1 I_1 + \xi_2 I_2$, where

$$\xi_1 := a_1 x + b_1 y + c_1 z, \quad \xi_2 := a_2 x + b_2 y + c_2 z.$$

A domain $\Omega \subset E_3$ we identify with the domain $\{(x, y, z) \in \mathbb{R}^3 : \zeta \in E_3\}$.

A locally bounded mapping $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ is called *right-G-monogenic* in the domain $\Omega \subset E_3$, if for every point $\zeta \in \Omega$ there exists an element $\Phi'_r(\zeta)$ of the algebra $\widetilde{\mathbb{A}}_2$ such that the following equality is true:

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h\Phi'_r(\zeta) \quad \forall\, h \in E_3.$$

Theorem 1. [1] *Let a domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight lines $\xi_1(x, y, z) = 0$, $\xi_2(x, y, z) = 0$. Then every right-G-monogenic mapping $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ is of the form*

$$\Phi(\zeta) = F_1(\xi_1) I_1 + F_2(\xi_2) I_2 + F_3(\xi_2)\rho,$$

where F_1 is some holomorphic in the domain $D_1 := \{\xi_1 = a_1 x + b_1 y + c_1 z : \zeta \in \Omega\}$ function of the variable ξ_1, and F_2, F_3 are some holomorphic in the domain $D_2 := \{\xi_2 = a_2 x + b_2 y + c_2 z : \zeta \in \Omega\}$ functions of the variable ξ_2.

Let Ω be a domain in the algebra $\widetilde{\mathbb{A}}_2$. Further a domain $\Omega = \{\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3 \rho \in \widetilde{\mathbb{A}}_2\}$ we will be identify with the congruent domain in $\mathbb{C}^3$.

A function

$$\Phi(\zeta) = f_1(\xi_1, \xi_2, \xi_3) I_1 + f_2(\xi_1, \xi_2, \xi_3) I_2 + f_3(\xi_1, \xi_2, \xi_3)\rho, \tag{2}$$

where $f_k(\xi_1, \xi_2, \xi_3)$, $k = 1, 2, 3$, are holomorphic functions of three complex variables, is called *right–differentiable* in a domain Ω, if at every point $\zeta \in \Omega$ there exists the element $\Phi'(\zeta)$ of the algebra $\widetilde{\mathbb{A}}_2$ such that the equality $d\Phi = d\zeta \cdot \Phi'(\zeta)$ is fulfilled.

Theorem 2. [2] *A function* $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ *of the form* (2), *where* $f_k : \Omega \to \mathbb{C}^3$ *are holomorphic functions, is right–differentiable in the domain* $\Omega \subset \widetilde{\mathbb{A}}_2$ *if and only if the conditions*

$$\frac{\partial f_1}{\partial \xi_2} = \frac{\partial f_1}{\partial \xi_3} = 0, \quad \frac{\partial f_2}{\partial \xi_1} = \frac{\partial f_2}{\partial \xi_3} = 0, \quad \frac{\partial f_3}{\partial \xi_1} = 0, \qquad \frac{\partial f_1}{\partial \xi_1} = \frac{\partial f_3}{\partial \xi_3}$$

are satisfied.

Therefore, every right-differentiable function can be expressed in the form

$$\Phi(\zeta) = (\alpha\xi_1 + \beta)\, I_1 + f_2(\xi_2) I_2 + (f_3(\xi_2) + \alpha\xi_3)\, \rho, \quad \alpha, \beta \in \mathbb{C}.$$

A function $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ of the form (2) is called *H-analytic* in a domain Ω, if Φ is differentiable by Hausdorff at every point $\zeta \in \Omega$, i.e., if the differential $d\Phi$ is a $\mathbb{C}$-linear homogeneous polynomials of the differential $d\zeta = d\xi_1 I_1 + d\xi_2 I_2 + d\xi_3 \rho$:

$$d\Phi = \sum_{s=1}^{9} A_s(\zeta)\, d\zeta\, B_s(\zeta)\,,$$

where A_s and B_s are some $\widetilde{\mathbb{A}}_2$-valued functions of the variable ζ

Theorem 3. [3] *A function* $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ *of the form* (2), *where* $f_k : \Omega \to \mathbb{C}^3$ *are holomorphic functions, is H-analytic in the domain* $\Omega \subset \widetilde{\mathbb{A}}_2$ *if and only if the conditions*

$$\frac{\partial f_1}{\partial \xi_2} = \frac{\partial f_1}{\partial \xi_3} = \frac{\partial f_2}{\partial \xi_1} = \frac{\partial f_2}{\partial \xi_3} = 0, \quad \frac{\partial^2 f_3}{\partial \xi_3^2} = 0$$

are satisfied.

Therefore, every H-analytic function can be expressed in the form

$$\Phi(\zeta) = f_1(\xi_1) I_1 + f_2(\xi_2) I_2 + [\varphi_1(\xi_1, \xi_2)\xi_3 + \varphi_2(\xi_1, \xi_2)]\, \rho.$$

R E F E R E N C E S

1. Kuzmenko, T., Shpakivskyi, V. *G-monogenic mappings in a three-dimensional noncommutative algebra*, Complex Variables and Elliptic Equations, **67**(11), 2759 – 2769 (2022).

2. Kuzmenko, T., Shpakivskyi, V. *Differentiable functions in a three-dimensional associative noncommutative algebra*, Advances in the Theory of Nonlinear Analysis and Its Application, **6**(1), 66 – 73 (2023).

3. Shpakivskyi, V., Kuzmenko, T. *Hausdorff analytic functions in a three-dimensional associative noncommutative algebra*, Journal of Mathematical Sciences, **262**(2), 207 – 221 (2022).

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AN ANALOG OF TOLSTOV THEOREM IN A COMMUTATIVE THREE-DIMENSIONAL BANACH ALGEBRA

We weaken the conditions of monogenicity for functions taking values in commutative banach algebra $\mathbb{A}_3$ over the field of complex numbers by generalizing Tolstov's theorem.

Theorem [G. Tolstov]. *Let*

$$f(z) = u(x, y) + iv(x, y)$$

be a locally bounded function of the complex variable $z = x + iy$ in a domain $D \subset \mathbb{C}$. Suppose that the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

exist everywhere in D and satisfy the Cauchy–Riemann equations almost everywhere:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Then the function $f(z)$ is analytic in D.

Consider the three-dimensional commutative associative Banach algebra $\mathbb{A}_3$ with identity 1 over the field $\mathbb{C}$, whose basis is the triple $\{1, \rho, \rho^2\}$, where $\rho^3 = 0$. Note that this basis is a Cartan basis.

Define the Euclidean norm of an element of the algebra by

$$\|a + b\rho + c\rho^2\| := \sqrt{|a|^2 + |b|^2 + |c|^2}, \qquad a, b, c \in \mathbb{C}.$$

The algebra $\mathbb{A}_3$ has a unique maximal ideal

$$\mathcal{I} := \{\lambda_1\rho + \lambda_2\rho^2 : \lambda_1, \lambda_2 \in \mathbb{C}\},$$

which is also its radical.

Consider the linear mapping $f : \mathbb{A}_3 \to \mathbb{C}$ defined by

$$f(a + b\rho + c\rho^2) = a.$$

Since the kernel of f is the maximal ideal $\mathcal{I}$, the mapping f is a continuous multiplicative functional.

Fix a real three-dimensional subspace

$$E_3 = \{\zeta = xe_1 + ye_2 + ze_3 : x, y, z \in \mathbb{R}\} \subset \mathbb{A}_3,$$

where the vectors e_1, e_2, e_3 are linearly independent over the field of real numbers $\mathbb{R}$, but do not necessarily form a basis of the algebra $\mathbb{A}_3$. We impose only one condition on the choice of the subspace E_3:

$$f(E_3) = \mathbb{C},$$

that is, the image of E_3 under f is the entire complex plane.

Let $\Phi : \Omega \to \mathbb{A}_3$ be a function defined in a domain $\Omega \subset E_3$. If for every point $\zeta \in \Omega$ there exists an element $\Phi'_G(\zeta) \in \mathbb{A}_3$ such that

$$\lim_{\delta \to 0+0} \big(\Phi(\zeta + \delta h) - \Phi(\zeta)\big)\delta^{-1} = h\Phi'_G(\zeta) \qquad \forall h \in E_3, \tag{3}$$

then the function $\Phi'_G : \Omega \to \mathbb{A}_3$ is called the *Gâteaux derivative* of Φ.

A function $\Phi : \Omega \to \mathbb{A}_3$ is called *monogenic* in the domain $\Omega \subset E_3$ if Φ is continuous and has a Gâteaux derivative at every point of Ω

Theorem. *Let $\Omega \subset E_3 \subset \mathbb{A}_3$ be a domain whose intersections with lines parallel to the line $f^{-1}(0)$ are connected. Let $\Phi : \Omega \to \mathbb{A}_3$ be a locally bounded function satisfying* (3) *for the vectors*

$$\{\pm a,\ \pm b,\ \pm c\}.$$

Then the function Φ admits the representation

$$\Phi(\zeta) = \frac{1}{2\pi i}\int_\gamma \big(F_0(t) + F_1(t)\rho + F_2(t)\rho^2\big)(t-\zeta)^{-1}\,dt,$$

where F_0, F_1, and F_2 are analytic functions in the domain

$$D = f(\Omega),$$

and γ is a closed rectifiable Jordan curve in $D = f(\Omega) \subset \mathbb{C}$ enclosing the point $f(\zeta)$.

R E F E R E N C E S

1. Tolstov, G. P. On the curvilinear and iterated integral. *Trudy Mat. Inst. Steklov.* **35** (1950), 3–101.

2. Mel'nichenko, I. P. The representation of harmonic mappings by monogenic functions. *Ukr. Math. J.* **27** (1975), no. 5, 499–505.

3. Plaksa, S. A., and Shpakovskii, V. S. Constructive description of monogenic functions in a harmonic algebra of the third rank. *Ukr. Math. J.* **62** (2011), no. 8, 1251–1266.

4. Plaksa, S. A. On differentiable and monogenic functions in a harmonic algebra. *Zbirnyk Prats Instytutu Matematyky NAN Ukrainy* **14** (2017), no. 1, 210–221.

5. Tkachuk M. V. Tolstov‘s Theorem in the Commutative Banach Algebra A3 // E-print: arXiv:2605.16185 [math.CV], 2026, 9 pp.

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CAUCHY'S MEAN VALUE THEOREM IN COMMUTATIVE ALGEBRAS

The article [1] proves the following general Cauchy mean value theorem for complex-valued functions.

Theorem 1. *Suppose that D is a subset of the complex plane $\mathbb{C}$, and f and g are functions from D into $\mathbb{C}$, and a and b are different points in D such that:*

(1) *the line segment $[a, b]$ is contained in D;*

(2) *the restrictions of f and g to $[a, b]$ are continuous;*

(3) *the functions f and g are holomorphic at each point of (a, b).*

Then there are some points ξ_1 and ξ_2 in (a, b) such that

$$\mathrm{Re}\Big((f(b) - f(a))\, g'(\xi_1) \Big) = \mathrm{Re}\Big(f'(\xi_1)\, (g(b) - g(a)) \Big),$$

$$\mathrm{Im}\Big((f(b) - f(a))\, g'(\xi_2) \Big) = \mathrm{Im}\Big(f'(\xi_2)\, (g(b) - g(a)) \Big).$$

In this paper, we prove an analogue of the Cauchy mean value Theorem 1 for commutative associative algebras. The results of this work are published in article [2].

Let $\mathbb{A}$ be a n-dimensional $(1 \le n < \infty)$ commutative associative algebra with a basis $\{I_r\}_{r=1}^{n}$ over the complex field $\mathbb{C}$.

Let us consider the vectors $e_1 \equiv 1, e_2, \ldots, e_k$ in $\mathbb{A}$, where $2 \le k \le 2n$, and these vectors are linearly independent over the field of real numbers $\mathbb{R}$. It means that the equality

$$\sum_{r=1}^{k} \alpha_r e_r = 0, \qquad \alpha_r \in \mathbb{R},$$

holds if and only if $\alpha_r = 0$ for all $r = 1, 2, \ldots, k$.

Let $\zeta := \sum\limits_{r=1}^{k} x_r\, e_r$, where $x_r \in \mathbb{R}$. Denote by $E_k := \left\{ \zeta = \sum\limits_{r=1}^{k} x_r e_r : \ x_r \in \mathbb{R} \right\}$ the linear span of vectors $e_1 = 1, e_2, \ldots, e_k$ over the field $\mathbb{R}$.

Let Ω be a domain in E_k.

Definition 1. *A continuous function $\Phi : \Omega \to \mathbb{A}$ is monogenic in Ω if Φ is differentiable in the sense of Gateaux–Mel'nychenko in every point of Ω, i. e. if for every $\zeta \in \Omega$ there exists an element $\Phi'(\zeta) \in \mathbb{A}$ such that*

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h\Phi'(\zeta) \quad \forall\, h \in E_k. \tag{4}$$

$\Phi'(\zeta)$ is the Gateaux–Mel'nychenko derivative of the function Φ at the point ζ.

Setting in (4) $h = e_r$, $r = 1, 2, \ldots, k$, we get

$$\frac{\partial \Phi}{\partial x_r} = \Phi'(\zeta) e_r \qquad \text{for all} \quad r = 1, 2, \ldots, k. \tag{5}$$

Thus, if a function is monogenic, then equalities (5) are true.

The theory of monogenic functions is described in monograph [3].

Definition 2. *For the element* $c := \sum\limits_{s=1}^{n} c_s I_s$, $c_s \in \mathbb{C}$, *of the algebra* $\mathbb{A}$ *we denote*

$$\mathrm{Re}_s(c) := \mathrm{Re}(c_s), \qquad \mathrm{Im}_s(c) := \mathrm{Im}(c_s), \qquad s = 1, 2, \ldots, n.$$

Theorem 2. *Let* Ω *be a domain in* $E_k \subset \mathbb{A}$, *and let* Φ *and* Ψ *be functions from* Ω *into* $\mathbb{A}$, *and* a *and* b *are different points in* Ω *such that:*

(1) *the line segment* $[a, b]$ *is contained in* Ω*;*

(2) *the element* $(b - a)$ *is invertible;*

(3) *the restrictions of* Φ *and* Ψ *to* $[a, b]$ *are continuous;*

(4) *the functions* Φ *and* Ψ *are* $\mathbb{R}$*-differentiable and monogenic at each point of* (a, b).

Then, on (a, b) *there exist some points* ξ_s , $s = 1, 2, \ldots, n$, *such that*

$$\mathrm{Re}_s\Big(\left(\Phi(b) - \Phi(a)\right) \Psi'(\xi_s)\Big) = \mathrm{Re}_s\Big(\left(\Psi(b) - \Psi(a)\right) \Phi'(\xi_s)\Big),$$

$$\mathrm{Im}_s\Big(\left(\Phi(b) - \Phi(a)\right) \Psi'(\xi_s)\Big) = \mathrm{Im}_s\Big(\left(\Psi(b) - \Psi(a)\right) \Phi'(\xi_s)\Big)$$

for all $s = 1, 2, \ldots, n$.

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R E F E R E N C E S

1. Száz, A. *A Cauchy's mean value theorem for complex functions*, Math. Student **64**, 125–127 (1995).

2. Shpakivskyi, V. S. *Analogues of Fermat's extreme value theorem and Cauchy's mean value theorem in commutative algebras*, Researches in Mathematics **34**(1), 178–187 (2026).

3. Plaksa, S. A., Shpakivskyi, V. S. Monogenic Functions in Spaces with Commutative Multiplication and Applications. Frontiers in Mathematics, Birkhäuser Cham, 550 p. (2023).

Секція 2
Екстремальні задачі / Extremal Problems

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VARIATIONAL EIGENVALUES ON METRIC MEASURE SPACES

The talk is devoted to the geometric analysis of Sobolev spaces on doubling metric measure spaces $X = (X, \rho, \mu)$ satisfying

$$\mu(B(x, 2r)) \leq C_\mu \, \mu(B(x, r)), \quad x \in X, \ r > 0,$$

and supporting a weak p-Poincaré inequality. In bounded Sobolev (p, p^*)-embedding domains $\Omega \subset X$, $p^* = \nu p/(\nu - p)$, $\nu = \log_2 C_\mu$, we establish the existence of minimizers for the variational Neumann (p, q)-eigenvalue problem, $1 < q < p^*$, and give a min–max characterization of the first non-trivial eigenvalue $\lambda_{p,q}(\Omega)$ in terms of the best constant in the corresponding (p, q)-Sobolev-Poincaré inequality.

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MOSAIC SYSTEM OF POINTS

Let the numbers $n, m, d \in \mathbb{N}$ be fixed. The system of points $A_{n,m} = \{a_{k,p} \in \mathbb{C} : k = \overline{1,n}, p = \overline{1,m}\}$, is called an (n,m)-ray system of points, if, for all $k = \overline{1,n}$, the following relations hold:

$$0 < |a_{k,1}| < \ldots < |a_{k,m}| < \infty;$$

$$\arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,m} =: \theta_k;$$

$$0 = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} := 2\pi.$$

For such systems of points, let us consider the following quantities:

$$\alpha_k = \frac{1}{\pi}\left[\theta_{k+1} - \theta_k\right], \ k = \overline{1,n}, \ \alpha_{n+1} := \alpha_1, \ \alpha_0 := \alpha_n, \sum_{k=1}^{n} \alpha_k = 2.$$

Let $r(B,a)$ the define of inner radius domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ [1]. Consider the system of angular domains:

$$P_k\left(A_{n,m}\right) = \{w \in \mathbb{C} : \theta_k < \arg w < \theta_{k+1}\}, \quad k = \overline{1,n}.$$

For a fixed number $\beta, R \in \mathbb{R}^+$, $0 < \beta < \frac{2\pi}{n}$, consider the unique branch of the multibranch analytic function

$$z_k(w) = \frac{i}{R^{\frac{1}{\alpha_k}} \cdot \sin\frac{\beta}{\alpha_k}} \cdot \left(-\left(e^{-i\theta_k} w\right)^{\frac{1}{\alpha_k}} + R^{\frac{1}{\alpha_k}} \cdot \cos\frac{\beta}{\alpha_k}\right). \tag{6}$$

For each $k = \overline{1,n}$, it realizes the one-sheet conformal mapping of the domain P_k onto the right half-plane $\mathrm{Re} z > 0$.

For each $k = \overline{1,n}$ we denote

$$\begin{array}{l} \Omega_j^{(k)} := \left\{z : \ |z - i\varrho_j| = r_j, 0 \le \arg z \le \frac{\pi}{2}, \varrho_j \in \mathbb{R}, r_j \in \mathbb{R}^+\right\}, j = \overline{1,m}, \\ \Omega_j^{(k)} := \left\{z : \ |z - i\varrho_j| = r_j, -\frac{\pi}{2} \le \arg z \le 0, \varrho_j \in \mathbb{R}, r_j \in \mathbb{R}^+\right\}, \\ j = \overline{m+1, 2m}, \end{array} \tag{7}$$

where

$$\varrho_1 + r_1 > \varrho_2 + r_2 > \ldots > \varrho_{2m} + r_{2m}.$$

Let $\{b_k\}_{k=1}^n \subset \mathbb{C}$ be a set of points such that

$$b_k \in P_k, \ \arg b_k - \theta_k = \beta, \ |b_k| = R, \ k = \overline{1,n}.$$

Let, for each fixed k, $k = \overline{1,n}$, $\left\{L_j^{(k)}\right\}_{j=1}^{2m}$ – be a collection of curves such that

$$\begin{array}{l} L_j^{(k)} \subset \overline{P_k}, \ b_k \in L_j^{(k)}, \ j = \overline{1,2m}, \\ a_{k,p} \in L_{m-p+1}^{(k-1)}, \ a_{k,p} \in L_{m+p}^{(k)}, \ p = \overline{1,m}, \\ z_k : \ L_j^{(k)} \to \Omega_j^{(k)}, \ j = \overline{1,2m}. \end{array} \tag{8}$$

It is easy to see from relations (6), (7), (8) that

$$z_k\left(b_k\right) = 1, \ z_k\left(a_{k+1,p}\right) = i\lambda_p, \ z_k\left(a_{k,p}\right) = -i\lambda_{m+p},$$

$$a_{n+1,p} := a_{1,p},\ \lambda_t > 0, t = \overline{1,2m}, k = \overline{1,n}, p = \overline{1,m}.$$

For each $k = \overline{1,n}$ we denote the corresponding systems of points by

$$D^{(k)}_{2m,d} = \left\{ c^{(k)}_{j,s} \in L^{(k)}_j : 0 < \left| \arg z_k \left(c^{(k)}_{j,1} \right) \right| < \left| \arg z_k \left(c^{(k)}_{j,2} \right) \right| < \right.$$

$$\left. < \ldots < \left| \arg z_k \left(c^{(k)}_{j,d} \right) \right| < \frac{\pi}{2},\ j = \overline{1,2m},\ s = \overline{1,d} \right\}.$$

The system of points

$$AD_{n,m,d} = \bigcup_{k=1}^{n} D^{(k)}_{2m,d} \bigcup A_{n,m}$$

will be called mosaic.

Let $\left\{ B_{k,p}, B^{(k)}_{j,s} \right\}$ be any collection of pairwise nonoverlapping domains of the mosaic system $AD_{n,m,d}$ such that

$$\begin{array}{c} a_{k,p} \in B_{k,p},\ c^{(k)}_{j,s} \in B^{(k)}_{j,s},\ B_{k,p}, B^{(k)}_{j,s} \subset \overline{\mathbb{C}}, \\ k = \overline{1,n}, p = \overline{1,m}, j = \overline{1,2m}, s = \overline{1,d}. \end{array}$$

Let D, $D \subset \overline{\mathbb{C}}$, be any open set, and let the point $w = a \in D$. By $D(a)$, we denote a connected component of the set D, which contains the point a. For an arbitrary system of points $A_{n,m} = \{a_{k,p} \in \mathbb{C} : k = \overline{1,n}, p = \overline{1,m}\}$ and for the open set D, $A_{n,m} \subset D$ we denote, by $D_k\left(a_{s,p}\right)$, the connected component of the set $D\left(a_{s,p}\right) \bigcap \overline{P_k}$ containing the point $a_{s,p}$, $k = \overline{1,n}$, $s = k, k+1$, $a_{n+1,p} := a_{1,p}$.

We say that the open set D, $A_{n,m} \subset D$ satisfies the condition of disjointness relative to the system of points $A_{n,m}$ if the relation

$$\left[D_k(a_{k,t}) \bigcap D_k(a_{k,u}) \right] \bigcup \left[D_k(a_{k+1,t}) \bigcap D_k(a_{k+1,u}) \right] \bigcup$$

$$\bigcup \left[D_k(a_{k,t}) \bigcap D_k(a_{k+1,p}) \right] \bigcup$$

$$\bigcup \left[B^{(k)}_{j,s} \bigcap D_k(a_{k,p}) \right] \bigcup \left[B^{(k)}_{j,s} \bigcap D_k(a_{k+1,p}) \right] = \varnothing,$$

$k = \overline{1,n}$, $p, t, u = \overline{1,m}$, $t \neq u$ $j = \overline{1,2m}$, $s = \overline{1,d}$, $a_{n+1,p} := a_{1,p}$ on all angles $\overline{P_k}$, holds.

The author has obtained an upper bound for the functional

$$\prod_{k=1}^{n} \left(\prod_{p=1}^{m} r\left(D, a_{k,p}\right) \cdot \prod_{j=1}^{2m} \prod_{s=1}^{d} r\left(B^{(k)}_{j,s}, c^{(k)}_{j,s} \right) \right)$$

REFERENCES

1. A.K. Bakhtin, G.P. Bakhtina, Yu.B. Zelinskii *Topological-algebraic structures and geometric methods in complex analysis*, Inst. Math. NAS Ukraine, Kiev, (2008).

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ON THE EQUALITY OF CHARACTERISTICS OF APPROXIMATION IN ISOMETRIC FUNCTION SPACES BY ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

Let X^n be one of the spaces $C^n, L^n_\infty, L^n_{\bar{p}}, \hat{L}^n_{\bar{p}}$ $(\bar{1} \le \bar{p} < \bar{\infty})$ [1],

$$F^{X^n}_{\bar{\sigma}} = \{T_{\bar{\sigma}^n}(z) = T_{\bar{\sigma}^n}(x_1 + it_1, \dots, x_n + it_n) : T_{\bar{\sigma}^n}(x) \in X^n\}$$

is the space of entire functions of exponential type not larger than $\bar{\sigma}^n = (\sigma_1, \dots, \sigma_n)$, which belong to the space X^n on the real n-dimensional Euclidean space E^n, $\left(F^{L^n_p}_{\bar{\sigma}} = F^{p^n}_{\bar{\sigma}},\, F^{L^n_{\bar{p}}}_{\bar{\sigma}} = F^{\bar{p}^n}_{\bar{\sigma}},\, F^{\hat{L}^n_{\bar{p}}}_{\bar{\sigma}} = F^{\hat{\bar{p}}^n}_{\bar{\sigma}},\, F^{L^n_\infty}_{\bar{\sigma}} = F^{\infty^n}_{\bar{\sigma}}\right)$, $X^n \supset F^{X^n}_{\bar{\sigma}}(E^n) = \left\{T_{\bar{\sigma}^n}(x) : T_{\bar{\sigma}^n}(z) \in F^{X^n}_{\bar{\sigma}}\right\}$ is their restriction to the space E^n.

Let's denote by $U_{\bar{\sigma}^n}(\Lambda, f, x) = (\hat{\lambda}_{\bar{\sigma}^n} * f)(x) = \int\limits_{E^n} \hat{\lambda}_{\bar{\sigma}^n}(x - t) f(t)\, dt$ a linear operator defined by convolution with the kernel $\hat{\lambda}_{\bar{\sigma}^n}(t)$.

We shall determine sufficient conditions under which the operator $U_{\bar{\sigma}^n}(\Lambda, f, x)$ maps the space X^n into the subspace $F^{X^n}_{\bar{\sigma}}(E^n)$.

Theorem 1. *Let X^n be one of the spaces C^n, L^n_∞, $L^n_{\bar{p}}$, or L^n_p, where $1 \le p < \infty$ and $(\bar{1} \le \bar{p} < \bar{\infty})$. Let $\hat{\lambda}_{\bar{\sigma}^n}(z)$ be an entire functions of exponential type not larger than $\bar{\sigma}^n$, belonging to the space $F^{1^n}_{\bar{\sigma}}$, and let $f \in X^n$. Then the function*

$$U_{\bar{\sigma}^n}(\Lambda, f, z) = (\hat{\lambda}_{\bar{\sigma}^n} * f)(z) = \int\limits_{E^n} \hat{\lambda}_{\bar{\sigma}^n}(z - t) f(t)\, dt$$

is an entire functions of exponential type not larger than $\bar{\sigma}^n$, which belongs to the space X^n, i.e., $U_{\bar{\sigma}^n}(\Lambda, f, z) \in F^{X^n}_{\bar{\sigma}}$.

Theorem 2. *Let*

$$\lambda(u) = \prod_{j=1}^{n} \lambda^j(u_j) = \prod_{j=1}^{n} \left(a^j(u_j) + ib^j(u_j)\right),$$

where the functions $a^j(u_j)$ are real and even, and $b^j(u_j)$ are real and odd, and all are absolutely continuous on the segments $[-\sigma_j, \sigma_j]$ for $j = 1, \dots, n$, with the boundary conditions $a^j(|\sigma_j|) = b^j(|\sigma_j|) = 0$. Then for any $p > 1$, the function

$$\hat{\lambda}_{\bar{\sigma}^n}(z) = \frac{1}{(2\pi)^n} \int\limits_{\Delta^n_{\bar{\sigma}}} \lambda(u) e^{-iuz}\, du$$

belongs to the space $F^{p^n}_{\bar{\sigma}}$. Moreover, if the derivatives $(a^j(u_j))'$ and $(b^j(u_j))'$ belong to the space $L_{q(-\sigma_j, \sigma_j)}$ for some $q > 1$, then $\hat{\lambda}_{\bar{\sigma}^n}(z) \in F^{1^n}_{\bar{\sigma}}$.

Let $\Psi_{\bar{y}^m}(x)$ be a non-negative delta-like kernel. If X is either one of the spaces L_p $(1 \leq p < \infty)$ or the space C_r of uniformly continuous functions defined on E, then the space $X * \Psi_{\bar{y}^m}$ is isometric to the space X, and the subspace $F_\sigma^X(E)$ is isometric to the subspace $F_\sigma^X(E) * \Psi_{\bar{y}^m}$. Then, due to this isometry, the following equalities hold:

$$\rho_X(f, U_\sigma(\Lambda, f)) = \rho_{XM_m}(f * \Psi_{\bar{y}^m}, U_\sigma(\Lambda * \Psi_{\bar{y}^m}, f)),$$

$$\rho_X(M, U_\sigma(\Lambda, M)) = \rho_{XM_m}(\{M * \Psi_{\bar{y}^m}\}, U_\sigma(\Lambda, \{\Psi_{\bar{y}^m} * M\})),$$

$$E_\sigma(f)_X = E_\sigma(f * \Psi_{\bar{y}^m})_{XM_m}, \quad E_\sigma(M)_X = E_\sigma(\{M * \Psi_{\bar{y}^m}\})_{XM_m},$$

$$\lambda(M, F_\sigma^X(E))_X = \lambda(\{M * \Psi_{\bar{y}^m}\}, \{F_\sigma^X(E) * \Psi_{\bar{y}^m}\})_{XM_m}.$$

(The definitions of the approximation characteristics can be found in [1]).

R E F E R E N C E S

1. Bushev, D., Kal'chuk, I., Kharkevych, Y. *Approximation of isometric function spaces by entire functions of exponential type*, Carpathian Mathematical Publications. **17**(2), 679–692 (2025).

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EXCEPTIONAL FAMILIES OF MEASURES ON CARNOT GROUPS

We found necessary and sufficient condition for the family of intrinsic Lipschitz surfaces passing through a common point to be p-exceptional for $p \geq 1$.

Let $\mathbb{G}$ be a Carnot group, N its topological dimension, Q its Hausdorff dimension. A subgroup $\mathbb{M}$ of a Carnot group $\mathbb{G}$ is called a homogeneous subgroup if $\mathbb{M}$ is a homogeneous group with respect to the group dilation. We assume that the group has a decomposition $\mathbb{G} = \mathbb{M}_U \cdot \mathbb{H}_U$ into complementary homogeneous subgroups over a set $U \subset \mathbb{G}$. The decomposition may not be unique and is parametrized by a parameter $\alpha \in \mathbb{N}$.

Definition 1. Suppose $1 \leq \mathbf{d_t} \leq N-1$ and $1 \leq \mathbf{d_m} \leq Q-1$. A non-empty subset $S \subset \mathbb{G}$ is called an intrinsic $(\mathbf{d_t}, \mathbf{d_m})$-Lipschitz surface in $\mathbb{G}$ (having topological dimension $\mathbf{d_t}$ and the Hausdorff dimension $\mathbf{d_m}$) if to every point $x \in S$ there correspond an open neighborhood $U(x,r) \subset \mathbb{G}$, a decomposition $\mathbb{G} = \mathbb{M}_U \cdot \mathbb{H}_U$ and an open set $\Omega \subset \mathbb{M}_U$ such that

- $x \in U$;
- $\mathbb{M}_U$ is a $(\mathbf{d_t}, \mathbf{d_m})$-subgroup of $\mathbb{G}$;
- there exists an intrinsic Lipschitz map $f_U \colon \Omega \to \mathbb{H}_U$ such that $S \cap U = \mathrm{graph} f_U \cap U(x,r)$.

Theorem 1. *Let Σ be a family of intrinsic $(\mathbf{d_t}, \mathbf{d_m})$-Lipschitz surfaces where each graph is parametrized over one of the decompositions $\mathbb{G} = \mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$, $\alpha = 1, 2, \ldots, l$. Moreover any term $\mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$ is an element in the orbit of a subgroup K_α of the isometry group of $\mathbb{G}$ preserving the decomposition $\mathbb{M}_\alpha \cdot \mathbb{H}_\alpha$.*

Let $\Sigma_{\mathcal{U}} = S \cap \mathcal{U}$, $S \in \Sigma$, and $\mathcal{U} \subset \mathbb{G}$ be an open set. Then $M_p(\mathbf{E}_{\mathcal{U}}) = 0$ for $p \in (0,1)$.

Theorem 2. *Let $\Sigma \subset \Sigma^{(\mathbf{d_t}, \mathbf{d_m})}$ be a collection of intrinsic $(\mathbf{d_t}, \mathbf{d_m})$-Lipschitz graphs. Suppose that all the graphs $S \in \Sigma$ contain a common point $g_0 \in \mathbb{G}$. Then for $\mathbf{d_m} p \leq Q$ we have $M_p(\Sigma) = 0$.*

R E F E R E N C E S

1. Franchi, B., Serapioni, R. P. *Intrinsic Lipschitz graphs within Carnot groups*, Journal of Geometric Analysis (2016), no. 3, 1946–1994.

2. Franchi, B., Markina, I. *Exceptional families of measures on Carnot groups*, Analysis and Geometry in Metric Spaces, 11 (2023), no. 1, paper No 20220148, 37 pages.

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EVOLUTIONARY-TYPE ESTIMATES FOR INNER RADII OF DOMAINS RELATIVE TO N-RADIAL SYSTEMS OF POINTS

A function $g_B(z, a)$ which is continuous in $\overline{\mathbb{C}}$, harmonic in $B\backslash\{a\}$ apart from z, vanishes outside B, and in the neighborhood of a has the following asymptotic expansion

$$g_B(z, a) = -\ln|z-a| + \delta + o(1), \quad z \to a,$$

(if $a = \infty$, then $g_B(z, \infty) = \ln|z| + \delta + o(1)$, $z \to \infty$) is called the (classical) Green function of the domain B with pole at $a \in B$. The inner radius $r(B, a)$ of the domain B with respect to a point a is the quantity $r(B, a) = e^{\delta}$.

Let $n \in \mathbb{N}$. A set of points $A_n := \left\{a_k \in \mathbb{C} : k = \overline{1, n}\right\}$ is called n-radial system, if $|a_k| \in \mathbb{R}^+$, $k = \overline{1, n}$, and $0 = \arg a_1 < \arg a_2 < \ldots < \arg a_n < 2\pi$.

Let the function f be regular in the disk $U = \{z : |z| < 1\}$ and take values w therein belonging to a domain D that has an inner radius $r(D, a)$ at the point $a \in D$. If $f(0) = a$, then the following (Hayman's) inequality holds:

$$|f'(0)| \leqslant r(D, a). \tag{9}$$

Equality in (9) is attained if f maps the disk U conformally and univalently onto D; in this case, $r(D, a)$ is called the conformal radius of the domain D at the point a. The conformal radius of the domain D with respect to the point at infinity is defined as $r(D, \infty) = r(\varphi(D), 0)$, where $\varphi(z) = 1/z$.

This work is devoted to obtaining inequalities of evolutionary type for functionals of the following form for all values of the parameter $\gamma \in (0, n]$

$$I_n(\gamma) = r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$J_n(\gamma) = \left(r\left(B_0, 0\right) r\left(B_{\infty}, \infty\right)\right)^{\gamma} \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$I_n(\gamma_1, \ldots, \gamma_n) = \prod_{k=1}^{n} r^{\gamma_k}(B_k, a_k),$$

where $n \in \mathbb{N}$, $A_n = \{a_k\}_{k=1}^{n}$ is an arbitrary fixed n-radial system of points in the complex plane $\mathbb{C}\backslash\{0\}$; B_0, B_{∞}, $\{B_k\}_{k=1}^{n}$ is an arbitrary system of mutually non-overlapping domains such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_{\infty} \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$. These functionals have been studied in many papers under various conditions, but at present, there are no methods for obtaining their exact maxima for all values of the parameter $\gamma \in (0, n]$. In this paper, we obtain evolutionary-type estimates for the products of inner radii of mutually non-overlapping multiply connected domains with respect to fixed n-radial systems of points of the complex plane. The problem of obtaining evolutionary-type inequalities for these functionals is a generalization of the results of M.A. Lavrentiev, G.M. Goluzin, Z. Nehari, J. Jenkins, P.M. Tamrazov, and N.A. Lebedev.

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\varepsilon > 0$, $\gamma \in (0, n]$, $\tau \in (0, \gamma)$. Then, for an arbitrary fixed n-radial system of points $A_n = \{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ and any set of mutually* *non-overlapping domains* B_k, $k = \overline{0,n}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, *such that* $r(B_k, a_k) \geqslant \varepsilon$, $k = \overline{1,n}$, *the following inequality holds:*

$$I_n(\gamma) \leqslant n^{-\frac{\gamma-\tau}{2}} \varepsilon^{\tau-\gamma} I_n(\tau) \prod_{k=1}^{n} |a_k|^{\frac{2(\gamma-\tau)}{n}}.$$

Theorem 2. *Let* $n \in \mathbb{N}$, $n \geqslant 2$, $\varepsilon > 0$, $\gamma \in (0,n]$, $\tau \in (0,\gamma)$. *Then, for an arbitrary fixed n-radial system of points* $A_n = \{a_k\}_{k=1}^n \in \mathbb{C} \backslash \{0\}$ *and any set of mutually non-overlapping domains* B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, *under the condition that* $r(B_k, a_k) \geqslant \varepsilon$, $k = \overline{1,n}$, *the following inequality holds:*

$$J_n(\gamma) \leqslant (n+1)^{-(\gamma-\tau)\frac{n+1}{n+2}} \varepsilon^{-\frac{2n(\gamma-\tau)}{n+2}} J_n(\tau) \prod_{k=1}^{n} |a_k|^{\frac{2(\gamma-\tau)}{n+2}}.$$

Theorem 3. *Let* $n \in \mathbb{N}$, $n \geqslant 3$, *and let* γ_k, θ_k, $k = \overline{1,n}$, *be some positive real numbers such that* $\gamma_k \geqslant \theta_k$ *for* $k = \overline{1,n}$. *Then, for an arbitrary fixed n-radial system of points* $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}$ *and any set of domains* $B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, *such that* $a_k \in B_k$, $k = \overline{1,n}$, $B_i \cap B_j = \emptyset$, $i \neq j$, *and under the condition that* $r(B_k, a_k) \geqslant \varepsilon$, $k = \overline{1,n}$, *for some* $\varepsilon > 0$, *the following inequality is true:*

$$I_n(\gamma_1, \ldots, \gamma_n) \leqslant \varepsilon^{-\sum\limits_{k=1}^{n}(\gamma_k-\theta_k)} (n-1)^{-\frac{1}{2}\sum\limits_{k=1}^{n}(\gamma_k-\theta_k)} \times$$
$$\times \prod_{i,j=1, i<j}^{n} |a_j - a_i|^{\frac{2}{n-1}(\gamma_i+\gamma_j-\theta_i-\theta_j)} \prod_{k=1}^{n} r^{\theta_k}(B_k, a_k).$$

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R E F E R E N C E S

1. Bakhtin, A.K., Denega, I.V. *Generalized M.A. Lavrentiev's inequality*, J. Math. Sci. **262**(2), 138-153 (2022).

2. Bakhtin, A.K., Zabolotnii, Y.V. *Estimation of the products of some powers of inner radii for multiconnected domains*, Ukr. Math. J. **73**(9), 1341-1358 (2022).

3. Denega, I., Zabolotnyi, Y. *Some extremal problems on the Riemannian sphere*, Carpathian Math. Publ. **16**(2), 593-605 (2024).

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APPROXIMATION PROPERTIES OF WEIERSTRASS INTEGRALS ON THE CLASS W^1H_ω

Let C be the space of 2π-periodic continuous functions with the norm $\|f\|_C = \max\limits_t |f(t)|$.

Let $\omega(t)$ be a fixed modulus of continuity. A function $f \in C$ belongs to the class H_ω if $|f(t_1) - f(t_2)| \le \omega(|t_1 - t_2|),\ t_1, t_2 \in \mathbb{R}$.

Denote by W^r, $r \in \mathbb{N}$, the set of 2π-periodic functions whose derivatives up to order $r-1$ are absolutely continuous and whose r-th derivative satisfies $|f^{(r)}(t)| \le 1$ for almost all t. Then $W^rH_\omega = \{f \in W^r : \ f^{(r)} \in H_\omega\},\ r \in \mathbb{N}$.

In this paper, we study approximation properties of the Weierstrass integrals

$$W_\delta(f;x) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x+t)\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\frac{k^2}{\delta}} \cos kt\right\} dt, \qquad \delta > 0.$$

The main problem is to investigate the asymptotic behavior, as $\delta \to \infty$, of the quantity

$$\mathcal{E}(W^1H_\omega; W_\delta)_C = \sup_{f \in W^1H_\omega} \|f(\cdot) - W_\delta(f;\cdot)\|_C.$$

The following result gives an asymptotic estimate for the class W^1H_ω [1].

Theorem 1. *For any modulus of continuity $\omega(t)$, as $\delta \to \infty$, the following asymptotic inequality holds:*

$$\mathcal{E}(W^1H_\omega; W_\delta)_C \le \frac{1}{2\sqrt{\delta}} \int\limits_0^{\pi\sqrt{\delta}} \omega\left(\frac{2t}{\sqrt{\delta}}\right)\left[\Phi\left(\frac{\pi\sqrt{\delta}}{2}\right) - \Phi\left(\frac{t}{2}\right)\right] dt +$$

$$+ O\left(e^{-\frac{\pi^2\delta}{4}} \omega\left(\frac{1}{\sqrt{\delta}}\right)\right),$$

where $\Phi(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-u^2}\, du$ is the probability integral. If $\omega(t)$ is a convex modulus of continuity, then this inequality becomes an asymptotic equality.

R E F E R E N C E S

1. Kal'chuk, I.V., Kharkevych, Yu.I. *Approximation properties of some singular integral on the classes of functions defined by the modulus of continuity*, Researches in Mathematics, **34**(1), 34–48 (2026).

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ПРО ОДНУ ЗАДАЧУ КОЛМОГОРОВА–НІКОЛЬСЬКОГО

Розглянемо клас Гельдера H^α функцій f, заданих на сегменті $[-1;1]$ і які задовольняють на цьому сегменті умову

$$|f(t_1) - f(t_2)| \leq |t_1 - t_2|^\alpha \quad \forall t_1, t_2 \in [-1, 1], \quad 0 < \alpha \leq 1 \tag{10}$$

Нехай $\hat{H}_0(t) = \sqrt{\dfrac{1}{\pi}}, \hat{H}_i(t) = \sqrt{\dfrac{2}{\pi}} \cos i \arccos t, i \in N,$ – ортонормована з вагою $\dfrac{1}{\sqrt{1-t^2}}$ на $[-1, 1]$ система поліномів Чебишова першого роду. Для кожної неперервної функції f позначимо $c_i = c_i(f) = \int\limits_1^{-1} \dfrac{f(z)\hat{H}_k(z)}{\sqrt{1-z^2}} dt$ послідовність коефіцієнтів Фур'є функції f за системою $\hat{H}_i(t)$ [1]. Нескінченний ряд $\sum\limits_{i=0}^{\infty} c_i \hat{H}_i$ називають рядом Фур'є-Чебишова функції f.

За допомогою множини $M = \mu_\rho(i)$ функцій натурального аргумента, залежних від дійсного параметра a, $0 \leq \rho < 1$, $\mu_\rho(0) = 1$, кожної неперервної на $[-1, 1]$ функції f поставимо у відповідність ряд

$$\sum_{i=0}^{\infty} \mu_\rho(k) c_i \hat{H}_i(2), \ 0 \leq \rho < 1. \tag{11}$$

Нехай ряд (11) при всіх $0 \leq \rho < 1$ буде рядом Фур'є- Чебишова деякої неперервної функції. Якщо $\mu_\rho(i) = \rho^i, \rho = 0, 1, 2, \ldots$, то цю функцію будемо позначати $A_\rho(f; t; H)$.

$$E(H^\alpha; A_\rho; t) = \sup_{f \in H^\alpha} |f(t) - A_\rho(f; t; H)| \tag{12}$$

в кожній точці $t \in [-1, 1]$ при $\rho \to 1-$, $0 < \alpha \leq 1$.

Якщо у явному вигляді знайдена функція $\phi(\rho) = \phi(\rho; t)$ така, що при $\rho \to 1-$

$$E(H^\alpha; A_\rho; t) = \phi(\rho; t) + o(\phi(\rho; t)), t \in [-1, 1],$$

то згідно [2] будемо казати, що розв'язана задача Колмогорова-Нікольського для інтегралів Пуассона-Чебишова A_ρ класу $H^\alpha, 0 < \alpha \leq 1$.

В сучасних прикладних дослідженнях особливий інтерес становлять задачі Колмогорова-Нікольського саме для операторів Пуассона–Чебишова $A_\rho(f)$ на класах функцій Гельдера. В роботі отримано розв'язок задачі Колмогорова-Нікольського (3).

R E F E R E N C E S

1. Paszkowski S. *Zastosowania numeryczne wielomianów i szeregów Czebyszewa. Warszawa*: Państwowe Wydawnictwo Naukowe, 1975. 480 s.

2. Степанець О. І. *Методи теорії наближень*: у 2 ч. Київ: Ін-т математики НАН України, 2002. Ч. 2. 468 с.

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DISCRETE EQUATIONS AND EXTREMAL PROBLEMS

For a general discrete dynamics on a Banach and Hilbert spaces we give necessary and sufficient conditions of the existence of bounded solutions under assumption that the homogeneous difference equation admits an discrete dichotomy on the semi-axes. We consider the so called resonance (critical) case when the uniqueness of solution is disturbed. We show that admissibility can be reformulated in the terms of generalized or pseudo-invertibility. As an application we consider the case when the corresponding dynamical system is e-trichotomy. Examples of the simplest discrete boundary value problems which have a strong generalized bounded solutions were considered (the set of such solutions with using of the notion of strong Moore-Penrose operator were represented). Part of the results is an extension of Palmer's lemma to the case of boundary value problems, the generating operators of which can have a non-closed set of values. Such problems also arise when solving extreme problems.

Statement of the problem

Consider the following equation

$$x_{n+1} = A_n x_n + h_n, n \in \mathbb{Z}, \tag{13}$$

with boundary condition

$$lx. = \alpha, \tag{14}$$

where $A_n : B \to B$ - is a set of bounded operators, from the Banach space B into itself. Assume that

$$A = (A_n)_{n\in\mathbb{Z}} \in l_\infty(\mathbb{Z}, \mathcal{L}(B)), h = (h_n)_{n\in\mathbb{Z}} \in l_\infty(\mathbb{Z}, B).$$

It means that

$$|||A||| = \sup_{n\in\mathbb{Z}} ||A_n|| < +\infty, |||h||| = \sup_{n\in\mathbb{Z}} ||h_n|| < +\infty,$$

$$l : l_\infty(\mathbb{Z}, B) \to B_1$$

is a linear and bounded operator which translates bounded solutions of (13) into the Banach space B_1, α is an element of Banach space B_1. It should be noted that an arbitrary solution of a homogeneous equation can be represented as: $x_m = \Phi(m, n)x_n, m \geq n$, where

$$\Phi(m, n) = \begin{cases} A_{m-1}A_{m-2}...A_n, \text{ if } m > n \\ I, \text{ if } m = n. \end{cases}$$

It is clear, that $\Phi(m, 0) = A_{m-1}A_{m-2}...A_0$. Also, we denote $U(m) := \Phi(m, 0)$ and $U(0) = I$.

Suppose that the homogeneous equation is exponentially dichotomous on the semiaxes $\mathbb{Z}_+$ and $\mathbb{Z}_-$ with projectors P and Q in the space B respectively, which means that there are projectors $P(P^2 = P)$ and $Q(Q^2 = Q)$, constants $k_{1,2} \geq 1$, $0 < \lambda_{1,2} < 1$ such that

$$||U(n)PU^{-1}(m)|| \leq k_1\lambda_1^{n-m}, n \geq m,$$
$$||U(n)(I - P)U^{-1}(m)|| \leq k_1\lambda_1^{m-n}, m \geq n$$

for arbitrary $m, n \in \mathbb{Z}_+$ (dichotomy on $\mathbb{Z}_+$);

$$||U(n)QU^{-1}(m)|| \leq k_2\lambda_2^{n-m}, n \geq m$$
$$||U(n)(I-Q)U^{-1}(m)|| \leq k_2\lambda_2^{m-n}, m \geq n$$

for arbitrary $m, n \in \mathbb{Z}_-$ (dichotomy on $\mathbb{Z}_-$).

For the solutions of the considered linear boundary-value problem we can obtain the following estimates.

Theorem 1. *We have the following estimates for the norm of the solution*

$$||x_n|| \leq k_1\lambda_1^n||\mathcal{P}_{N(D)}c|| + k_1\lambda_1^n||D^-||\left(\frac{k_1\lambda_1}{1-\lambda_1} + \frac{k_2\lambda_2}{1-\lambda_2}\right)|||h|||+$$
$$+k_1\frac{(1+\lambda_1-\lambda_1^n)}{1-\lambda_1}|||h|||, \quad n \geq 0,$$

$$||x_n|| \leq k_2\lambda_2^{-n}||\mathcal{P}_{N(D)}c|| + k_2\lambda_2^{-n}||D^-||\left(\frac{k_1\lambda_1}{1-\lambda_1} + \frac{k_2\lambda_2}{1-\lambda_2}\right)|||h|||+$$
$$+k_2\frac{1+\lambda_2-\lambda_2^{-n+1}}{1-\lambda_2}|||h|||, \quad n \leq 0.$$

From these inequalities follows the estimate

$$|||x||| \leq K||\mathcal{P}_{N(D)}c|| + K||D^-||\left(\frac{k_1\lambda_1}{1-\lambda_1} + \frac{k_2\lambda_2}{1-\lambda_2}\right)|||h||| + K\frac{1+\Lambda_1}{1-\Lambda_2}|||h|||,$$

where $K = \max\{k_1, k_2\}$, $\Lambda_1 = \max\{\lambda_1, \lambda_2\}$, $\Lambda_2 = \min\{\lambda_1, \lambda_2\}$. *Here* $D = P - I + Q$, $\mathcal{P}_{N(D)} = I - D^-D$.

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R E F E R E N C E S

1. Palmer K. Exponential Dichotomies, the Shadowing Lemma and Transversal Homoclinic Points, Dynamics Reported, 1, 265–306, 1988, doi.10.1007/978-3-322-96656-8-5.

2. Smale S. Differentiable dynamical systems, Bull. Amer. Math. Soc., 73, 747–817, 1967, doi.10.1090S0002-9904-1967-11798-1.

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APPROXIMATION OF CONTINUOUS FUNCTION BY REPEATED DE LA VALLÉE POUSSIN MEANS

Let $\mathrm{L}(\mathbb{T}^d)$ be the space of all integrable functions $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_d)$ defined on $\mathbb{T}^d = [-\pi, \pi]^d$, which are 2π-periodic in each variable. Let $f \in \mathrm{L}(\mathbb{T}^d)$. The *rectangular repeated de la Vallée Poussin means* are defined as the arithmetic means of the rectangular partial Fourier sums. For fixed vectors

$$\mathbf{n}, \mathbf{u}, \mathbf{v} \in \mathbb{N}^d, \qquad u_i + v_i \le n_i, \quad i \in \overline{\{1, d\}},$$

they are given by

$$\mathcal{V}^{(2)}_{\mathbf{n},\mathbf{u},\mathbf{v}}[f](\mathbf{x}) = \prod_{i=1}^{d} \frac{1}{u_i} \sum_{k_i = n_i - u_i}^{n_i - 1} \mathcal{V}_{\mathbf{k+1},\mathbf{v}}[f](\mathbf{x}),$$

where

$$\mathcal{V}_{\mathbf{n},\mathbf{v}}[f](\mathbf{x}) = \prod_{i=1}^{d} \frac{1}{v_i} \sum_{k_i = n_i - v_i}^{n_i - 1} S_{\mathbf{k}}[f](\mathbf{x}),$$

and $S_{\mathbf{k}}[f]$ is the rectangular partial sum of the Fourier series of f.

We obtain asymptotic formulas for the exact upper bounds of the deviations of the rectangular repeated de la Vallée Poussin means on multidimensional analogues $\mathrm{C}^{\psi}_{\beta,\infty}(\mathbb{T}^d)$ of function classes generated by a fixed sequence $\psi = (\psi(k))_{k=1}^{\infty}$ and an argument shift $\beta \in \mathbb{R}$ (see the definitions in [1, p. 33], [2,3]).

Let us denote by

$$\mathcal{E}\left(\mathrm{C}^{\psi}_{\beta,\infty}(\mathbb{T}^d); \mathcal{V}^{(2)}_{\mathbf{n},\mathbf{u},\mathbf{v}}\right) := \sup_{f \in \mathrm{C}^{\psi}_{\beta,\infty}(\mathbb{T}^d)} \left\| f(\cdot) - \mathcal{V}^{(2)}_{\mathbf{n},\mathbf{u},\mathbf{v}}(\cdot) \right\|_C .$$

Our main result is follows.

Theorem 1. *Let $(\psi_i(k_i))_{k_i=1}^{\infty} \in D_{q_i}$, $(\Psi_i(k_i))_{k_i=1}^{\infty} \in D_{Q_i}$, $\beta, \beta^* \in \mathbb{R}^d$, $\mathbf{u}, \mathbf{v} \in \mathbb{N}^d$, and let $u_i + v_i < n_i$, $i \in \overline{\{1, d\}}$. Then, as $n_i - u_i - v_i \to \infty$, $i \in \overline{\{1, d\}}$, the following asymptotic formula holds:*

$$\mathcal{E}\left(\mathrm{C}^{\psi}_{\beta,\infty}(\mathbb{T}^d); \mathcal{V}^{(2)}_{\mathbf{n},\mathbf{u},\mathbf{v}}\right) = \frac{8}{\pi^2} \sum_{i=1}^{d} \frac{\psi_i(n_i - u_i - v_i + 1)}{u_i v_i (1 + q_i)^3} \Pi\left(\frac{4 q_i}{(1+q_i)^2}; \frac{2\sqrt{q_i}}{1 + q_i}\right) +$$
$$+ O(1) \left(\sum_{i=1}^{d} \left(\frac{\psi(n_i - u_i - v_i)}{u_i v_i (n_i - u_i - v_i)(1 - q_i)^4} + \frac{\psi_i(n_i - u_i - v_i + 1)(q_i^{u_i} + q_i^{v_i})}{u_i v_i (1 - q_i)^3} + \right.\right.$$
$$\left.\left. + \frac{\varepsilon_{n_i - u_i - v_i + 1}(\psi_i)}{(1 - q_i)^2} \right) + \Pi^{\mathbf{Q}}_{\mathbf{n},\mathbf{u},\mathbf{v}}(\Psi_1; \ldots; \Psi_d) \right), \qquad (15)$$

where $\Pi(n; k)$ is the complete elliptic integral of the third kind,

$$\varepsilon_m(\psi) = \sup_{k \ge m} \left| \frac{\psi(k+1)}{\psi(k)} - q \right|, \qquad q = \lim_{k \to \infty} \frac{\psi(k+1)}{\psi(k)},$$

$$\Pi^{\mathbf{Q}}_{\mathbf{n},\mathbf{u},\mathbf{v}}\left(\Psi_1;\ldots;\Psi_d\right):=\sum_{r=2}^{d}\sum_{\mu(r)\subset\overline{\{1,d\}}}\sum_{\xi\subset\mu(r)}\prod_{s\in\mu}\frac{\Psi_s(n_s-p_s+1)}{u_s v_s\,(1-Q_s)^3}\prod_{j\in\xi}\frac{\varepsilon_{n_j-u_j-v_j}(\Psi_j)}{(1-Q_j)^3},$$

and $O(1)$ *is a quantity uniformly bounded with respect to all parameters.*

Formula (15) is an asymptotically exact equality under the conditions $u_i\to\infty$, $v_i\to\infty$, $q_i=Q_i$, and

$$\lim_{n_i\to\infty}\varepsilon_{n_i-u_i-v_i}u_i v_i=0,\qquad i\in\overline{\{1,d\}}.$$

REFERENCES

1. Novikov, O.O., Rovenska, O.G. *Approximation of Periodic Functions by Rectangular Linear Methods*, DSEA, Kramatorsk (2012).
2. Rovenska, O. *Approximation of differentiable functions by rectangular repeated de la Vallée Poussin means*, Journal of Analysis **31**(1), (2023).
3. Rovenska, O. *Approximation of classes of Poisson integrals by rectangular Fejér means*, Frontiers in Applied Mathematics and Statistics **10**, 1437247 (2024).

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ON THE BEST NON-SYMMETRIC RELATIVE APPROXIMATIONS OF SOME CONVOLUTION CLASSES BY GENERALIZED SPLINES

Let L_p $(1 \le p \le \infty)$ be the space of 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with standard norms $\|\cdot\|_p$. For $\alpha, \beta > 0$, the non-symmetric norm is defined by $\|f\|_{p;\alpha,\beta} = \|\alpha f_+ + \beta f_-\|_p$, where $f_\pm(t) = \max\{\pm f(t), 0\}$. The best non-symmetric (α,β)-approximation of a class $M \subset L_p$ by a set $H \subset L_p$ in L_p metric is defined as $E(M,H)_{p;\alpha,\beta} = \sup_{f\in M} \inf_{u\in H} \|f - u\|_{p;\alpha,\beta}$.

Let $0 < h < \frac{2\pi}{n}$. We consider generalized splines with floating knots $x_{2k} = \frac{2k\pi}{n}$ and $x_{2k+1} = \frac{2k\pi}{n} + h$ $(k \in \mathbb{Z})$. For a given kernel $K \in CVD$ $(K \perp 1)$, let $S^h_{2n,K}$ denote the set of generalized splines of the form

$$u(t) = c + \sum_{k=1}^{2n} c_k K(t - x_k), \quad c \in \mathbb{R}, \quad \sum_{k=1}^{2n} c_k = 0,$$

and $S^{1,h}_{2n,K}$ denote the set of generalized splines of the form

$$u(t) = c + \int_0^{2\pi} K(t - v) s_0(v) dv,$$

where $s_0 \in S^h_{2n,0}$ is a piecewise constant function with step knots at points x_k.

Let $\varphi_{1;\alpha,\beta}(t)$ be the 2π-periodic comparison function, and let $\varphi_{1,K;\alpha,\beta}(t) = (K * \varphi_{1;\alpha,\beta})(t)$. On the interval $[0,\pi]$, we define the functions:

$$\Phi_{1,K}(\alpha,\beta;x) = \frac{1}{2}\left(\varphi_{1,K;\alpha,\beta}(x) + \varphi_{1,K;\alpha,\beta}(\pi - x)\right),$$
$$R_{1,K}(\alpha,\beta;x) = P\left(\varphi_{1,K;\alpha,\beta}(x) + \varphi_{1,K;\alpha,\beta}(\pi - x)\right),$$

where $P(f,x)$ denotes the non-increasing rearrangement of the function f on $[0,\pi]$.

We study the problems of best non-symmetric L_1-approximation for the convolution classes $K_1 * W^0_1$ and $K_2 * W^0_1$ by sets of generalized splines $S^h_{2n,K_1} \cap K * W^0_V$ and $S^{1,h}_{2n,K_2} \cap K_2 * W^0_1$, respectively, where $K_r = K * B_r$ and B_r is the Bernoulli kernel.

Theorem 1. *For any $n \in \mathbb{N}$ and odd kernel $K \in CVD$, the following equality holds:*

$$E(K_1 * W^0_1, S^h_{2n,K_1} \cap K * W^0_V)_{1;\alpha,\beta} = \Phi_{1,K_1}(\alpha,\beta;t_{\max}) - \frac{1}{2}\left(\Phi_{1,K_1}\left(\alpha,\beta;t_{\max} + \frac{h}{2}\right) + \Phi_{1,K_1}\left(\alpha,\beta;t_{\max} + \frac{\pi}{n} - \frac{h}{2}\right)\right),$$

where $t_{\max}$ is the point such that $\varphi_{1,K_1;\alpha,\beta}(t_{\max}) = \max_{t\in\mathbb{R}} \varphi_{1,K_1;\alpha,\beta}(t)$.

Theorem 2. *For any $n \in \mathbb{N}$ and even kernel $K \in CVD$, the following equality holds:*

$$E(K_2 * W^0_1, S^{1,h}_{2n,K_2} \cap K_2 * W^0_1)_{1;\alpha,\beta} = \frac{1}{2} R_{1,K_2}(\alpha,\beta;t_{\max}) - \frac{M}{2}\int_{t_{\max}}^{t_{\max}+M} R_{1,K_2}(\alpha,\beta;t)dt,$$

where $M = \max\left\{\frac{1}{h}, \frac{1}{\frac{2\pi}{n} - h}\right\}$, and $t_{\max}$ is the maximum point of $\varphi_{1,K_2;\alpha,\beta}(t)$.

As corollaries, exact values for standard best one-sided approximations $E^\pm(\cdot)_1$ are also obtained.

Секція 3

Задачі математичної фізики / Problems of Mathematical Physics

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ЗБУРЕННЯ СИСТЕМИ РІВНЯНЬ ГАЗОВОЇ ДИНАМІКИ

Для системи диференціальних рівнянь з частинними похідними першого порядку розглянемо задачу Коші:

$$\begin{cases} \rho_t + (u\rho)_x = 0, \\ u_t + \left(\frac{u^2}{2} + \theta(\rho)\right)_x = 0, \end{cases} \tag{1}$$

$$\begin{cases} u|_{t=0} = u_0(x), \\ \rho|_{t=0} = \rho_0(x), \end{cases} \tag{2}$$

де $u = u(t,x)$, $\rho = \rho(t,x)$, $\theta(\rho) \in C^{2,\alpha}$, $\theta'\rho > 0$, $\theta''(\rho) > 0$, $u_0(x), \rho_0(x) \in L^\infty(\mathbb{R})$.

За виконаних умов система (1) для газодинамічних параметрів є гіперболічною.

Разом з задачею (1), (2) при кожному $\varepsilon > 0$ розглянемо задачу Коші для збуреної системи:

$$\begin{cases} \rho_t^\varepsilon + (u^\varepsilon \rho^\varepsilon)_x = \varepsilon \rho_{xx}^\varepsilon, \\ u_t^\varepsilon + \left(\frac{(u^\varepsilon)^2}{2} + \theta(\rho^\varepsilon)\right)_x = \varepsilon u_{xx}^\varepsilon, \end{cases} \tag{3}$$

$$\begin{cases} u^\varepsilon|_{t=0} = u_0^\varepsilon(x), \\ \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon(x). \end{cases} \tag{4}$$

Тут $u_0^\varepsilon(x), \rho_0^\varepsilon(x)$ — середні функції для функцій $u_0(x), \rho_0(x)$.

Нехай $U(\rho,u) = \frac{u^2}{2} + \theta(\rho)$, $F(\rho,u) = \frac{u^3}{3} + u\theta'\rho$. Ці функції утворюють ентропійну пару, яка дозволяє виділити допустимий узагальнений розв'язок задачі (1), (2).

Обмежена вимірна вектор-функція $(\rho(t,x), u(t,x))$ називається *узагальненим розв'язком* задачі (1), (2), якщо $\forall f(t,x) \in C_0^\infty([0,T] \times \mathbb{R}^1)$ виконуються рівності:

$$\int_0^T \int_{-\infty}^{+\infty} (f_t\rho + f_x u\rho)\, dx\, dt = 0,$$

$$\int_0^T \int_{-\infty}^{+\infty} \left(f_t u + f_x \left(\frac{u^2}{2} + \theta(\rho)\right)\right) dx\, dt = 0,$$

та $\forall f(t,x) \in C_0^\infty([0,T] \times \mathbb{R}^1)$, $f(t,x) \geq 0$ виконується нерівність:

$$-\int_0^T \int_{-\infty}^{+\infty} (U(\rho,u)f_t + F(\rho,u)f_x)\,dx\,dt \leq 0,$$

а початкові умови (2) приймаються у сильному сенсі в $L_{1,loc}(\mathbb{R})$.

Theorem 1. *Нехай $u_0(x), \rho_0(x) \in C^2$, тоді задача (1), (2) має гладкий розв'язок у деякому околі $t = 0$.*

У випадку довільних початкових функцій для системи (1) виникають розривні розв'язки задачі (1), (2). Один з підходів для визначення допустимих розв'язків полягає у переході до параболічної системи і апроксимації за допомогою розв'язків нової відповідної задачі.

Theorem 2. *Сім'я розв'язків $(\rho^\varepsilon, u^\varepsilon)$ задачі (3), (4) компактна, і для деякої послідовності ε_n маємо $(\rho^{\varepsilon_n}, u^{\varepsilon_n}) \to (\rho, u)$, де (ρ, u) — узагальнений розв'язок задачі (1), (2).*

Отримані результати є варіацією результатів робіт [1-2].

Л І Т Е Р А Т У Р А

1. Казмерчук, А.І. *До обґрунтування наближених методів розв'язання квазілінійних законів збереження з негладкими даними задачі*, Вісник національного університету "Львівська політехніка", Прикладна математика. **411**, 147–151 (2000).

2. Казмерчук, А.І. *Оптимізація швидкості збіжності в методах наближеного розв'язування задачі Коші для системи квазілінійних рівнянь з частинними похідними першого порядку*, Прикарпатський вісник НТШ. Число. **2(46)**, 47–51 (2018).

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A NOVEL HYBRID L1-DGJM APPROACH FOR FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING THE CAPUTO–FABRIZIO OPERATOR

In this work, a novel hybrid L1-DGJM scheme is proposed for solving fractional differential equations involving the Caputo–Fabrizio derivative operator. The proposed approach combines the L1 approximation for the discretization of the fractional derivative with the Daftardar–Gejji and Jafari Method (DGJM) for constructing the solution iteratively. We develop the numerical scheme for Caputo–Fabrizio fractional-order differential equations of the form

$$^{CF}D_t^{\eta}\psi(t) = f(t, \psi(t)), \qquad \psi(0) = \psi_0, \quad 0 < \eta < 1, \quad t \in [0, T],$$

where $^{CF}D_t^{\eta}$ represents the Caputo–Fabrizio fractional derivative. The stability and error analysis of the proposed method are established, and the approach is further extended to systems of fractional differential equations. The effectiveness and applicability of the proposed scheme are demonstrated through several numerical examples, including the fractional Riccati equation, Lotka–Volterra system, and Rikitake system. The obtained results are presented graphically and compared with the solutions generated by the fractional Adams–Bashforth method and a recently developed predictor–corrector method. The numerical results indicate that the proposed hybrid L1-DGJM scheme provides highly accurate and efficient solutions for Caputo–Fabrizio fractional differential equations and their systems.

R E F E R E N C E S

1. V. Daftardar-Gejji and H. Jafari, An iterative method for solving nonlinear functional equations, Journal of mathematical analysis and applications, vol. 316, no. 2, pp. 753–763, 2006..

2. M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, Progress in Fractional Differentiation and Applications, vol. 1, no. 2, pp. 73–85, 2015.

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ELECTRIC FIELD INFLUENCE ON OPTICAL RESPONSE OF ELLIPTICAL NANOTUBES

We investigate the electronic states and optical response of electrons confined in elliptical nanotubes under the action of an external electric field. The system is modeled within the effective mass approximation, where the Hamiltonian takes the form

$$\hat{H} = -\frac{\hbar^2}{2m^*}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + V(x,y) - e\mathbf{F}\cdot\mathbf{r}, \tag{16}$$

with m^* the effective mass, $V(x,y)$ the confining potential of the elliptic cross section, and $\mathbf{F}$ the applied electric field. The boundary conditions correspond to hard walls at the ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{17}$$

where a and b are the semi-axes of the ellipse. Numerical solutions were obtained using the finite element method, allowing us to calculate eigenvalues E_n and eigenfunctions $\psi_n(x,y)$.

The analysis shows that ellipticity strongly modifies the energy spectrum, leading to quasi-degenerate states and anticrossings when the field is applied along different axes. The oscillator strength of optical transitions is given by

$$f_{ij} \propto (E_j - E_i)\left|\langle\psi_i|\hat{x}|\psi_j\rangle\right|^2, \tag{18}$$

which demonstrates clear dependence on the orientation of the electric field. For fields along the major axis, transitions polarized in the x direction dominate, while fields along the minor axis enhance y-polarized transitions. This tunability of anisotropic optical response highlights the potential of elliptical nanotubes for polarization-sensitive optoelectronic devices.

Our results extend earlier studies of elliptic quantum wires [1] and theoretical models of electric field effects in elliptical systems [2], confirming that geometry and external fields act as complementary parameters for tailoring quantum confinement and optical properties.

R E F E R E N C E S

1. Holovatsky, V.A., Gutsul, V.I. *Electron energy spectrum and wave functions in complicated elliptic quantum wires*, J. Optoelectron. Adv. Mater. **9**, 1437–1441 (2007).

2. Holovatsky, V.A., Yarema, V.V., Holovatska, N.H. *Theory of electric field effect on the optical properties of elliptical quantum wire*, Mol. Cryst. Liq. Cryst. **768**, 729–736 (2024).

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ГЕОМЕТРИЧНА ІНТЕРПРЕТАЦІЯ ЛОРЕНЦ–ІНВАРІАНТНОЇ МОДИФІКАЦІЇ РІВНЯННЯ ШРЕДІНГЕРА

Проблема побудови узгодженого теоретичного опису фізичної реальності, що поєднує принципи квантової механіки та спеціальної теорії відносності (СТО), залишається однією з центральних у сучасній фундаментальній фізиці [1, 2]. Незважаючи на успіхи квантової теорії поля, питання про можливість формулювання рівнянь руху, які б одночасно задовольняли умовам щодо хвильової функції та були лоренц–інваріантними, не втратило своєї актуальності [3, 4].

Класичне рівняння Шредінгера не є інваріантним відносно перетворень Лоренца, що обмежує сферу його застосування [5]. Відомі релятивістські узагальнення — рівняння Клейна–Гордона та Дірака — забезпечують коваріантний опис, однак супроводжуються низкою концептуальних труднощів [4, 5].

У даній роботі запропоновано підхід до модифікації рівняння Шредінгера, що базується на його аналітичному перетворенні [4]. Розглянемо одновимірне рівняння для вільної частинки ($U = 0$):

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \tag{2}$$

Для виявлення прихованих симетрій використаємо допоміжне диференціальне рівняння [6]:

$$\frac{\partial^2\Psi}{\partial t^2} + i\omega\frac{\partial\Psi}{\partial t} = 0 \tag{5}$$

Припускаючи стаціонарний стан $\Psi(x,t) = R(x)e^{-i\frac{E_\omega}{\hbar}t}$, ми можемо виразити першу похідну через другу та, після введення координати $w = ct$, отримати [7, 8]:

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{2E_m}{E_\omega}\frac{\partial^2\Psi}{\partial w^2} = 0 \tag{15}$$

Розглянемо випадок $2E_m/E_\omega = 1$. Тоді рівняння (15) набуває вигляду оператора д'Аламбера:

$$\Box\Psi = \frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial w^2} = 0 \tag{21}$$

Доведемо лоренц-інваріантність отриманої форми. Перейдемо до системи x', що рухається зі швидкістю v: $x' = \gamma(x - vt)$, $w' = \gamma(w - \beta x)$. Використовуючи ланцюгове правило для частинних похідних [9]:

$$\frac{\partial}{\partial x} = \gamma\frac{\partial}{\partial x'} - \gamma\beta\frac{\partial}{\partial w'}, \quad \frac{\partial}{\partial w} = -\gamma\beta\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial w'} \tag{19}$$

можна показати, що $\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi}{\partial w^2} = \frac{\partial^2\Psi}{\partial x'^2} - \frac{\partial^2\Psi}{\partial w'^2}$ [10].

Для загального квантовомеханічного випадку введемо параметр $\alpha = E_\omega/2E_m$ [11]. Тоді узагальнене рівняння записується через масштабовану часову координату $w_\alpha = \alpha ct$ [12]:

$$\Box_\alpha\Psi(x, y, z, w_\alpha) = 0 \tag{42}$$

Фізичний зміст параметра α. Введений коефіцієнт визначає масштаб часової координати. Це можна трактувати як деформацію часової осі, де хвиля «бачить» інший масштаб часу залежно від її енергетичного стану [13, 14]:

- $\alpha = 1$ — релятивістський режим, де рівняння (42) описує поширення хвилі зі швидкістю c [15, 16].
- $\alpha < 1$ — квантовий режим, де енергія менша за енергію спокою, що призводить до «розтягування» часу [16].
- $\alpha > 1$ — надрелятивістський режим, що характеризується «стисненням» часу [16].

Така інтерпретація дозволяє не лише зберегти імовірнісний зміст хвильової функції, а й інтегрувати релятивістську симетрію безпосередньо в структуру рівняння Шредінгера [10, 14].

R E F E R E N C E S

1. Давидов О. С. *Квантова механіка: підручник.* Київ: ВД «Академперіодика», 2012. 708 с.

2. Dirac P. A. M. *The Principles of Quantum Mechanics.* 4th ed. Oxford: Clarendon Press, 1958. 312 p.

3. Griffiths D. J., Schroeter D. F. *Introduction to Quantum Mechanics.* 3rd ed. Cambridge: Cambridge University Press, 2018. 500 p.

4. Peskin M. E., Schroeder D. V. *An Introduction to Quantum Field Theory.* Boulder: Westview Press, 1995. 842 p.

5. Schrödinger E. Quantisierung als Eigenwertproblem. *Annalen der Physik.* 1926. Bd. 79. S. 361–376.

6. Стрижак П. Є. *Квантова хімія: підручник.* Київ: Києво-Могилянська академія, 2009. 458 с.

7. Вакарчук І. О. *Квантова механіка: підручник.* 3-тє вид., доп. Львів: ЛНУ імені Івана Франка, 2007. 848 с.

8. Висоцький В. І. *Квантова механіка та її використання у прикладній фізиці: підручник.* Київ: Київський університет, 2008. 367 с.

9. Weinberg S. *The Quantum Theory of Fields.* Vol. 1: Foundations. Cambridge: Cambridge University Press, 1995. 609 p.

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МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ ЕНЕРГЕТИЧНОГО СПЕКТРА ТА ХВИЛЬОВИХ ФУНКЦІЙ ЕЛЕКТРОНА У ГЕКСАГОНАЛЬНИХ НАНОГЕТЕРОСТРУКТУРАХ

Розвиток сучасної наноелектроніки вимагає точного математичного моделювання квантово-механічних процесів у напівпровідникових структурах зі складною геометрією. У цій роботі досліджується енергетичний спектр та просторовий розподіл хвильових функцій електрона в нанотрубці безмежної довжини з поперечним перерізом у формі правильного шестикутника [1]. Така система розглядається як багатошарова гетероструктура, де потенціальний рельєф визначається різницею спорідненості до електрона матеріалів внутрішнього циліндра, самої трубки та зовнішнього середовища [2, 3].

Основним завданням є знаходження власних значень та власних функцій оператора Гамільтона:

$$\hat{H}\Psi = E\Psi \tag{3.1}$$

У циліндричній системі координат (ρ, ϕ, z) гамільтоніан для електрона з координатно-залежною ефективною масою $m^*(\rho, \phi)$ має вигляд:

$$\hat{H} = -\frac{\hbar^2}{2}\nabla\left(\frac{1}{m^*(\rho, \phi)}\nabla\right) + U(\rho, \phi) \tag{3.2}$$

Потенціальна енергія $U(\rho, \phi)$ моделюється як кусково-стала функція, що набуває значень $-U_0, -U_1, -U_2$ у відповідних областях наноструктури [4, 5].

З огляду на трансляційну інваріантність системи вздовж аксіальної осі OZ, повну хвильову функцію можна представити у вигляді добутку плоскої хвилі та функції поперечного руху:

$$\Psi(\rho, \phi, z) = \frac{1}{\sqrt{L}}\psi(\rho, \phi)e^{ik_z z} \tag{3.5}$$

Підстановка цього виразу в основне рівняння дозволяє виділити енергію поздовжнього руху $E_z = \frac{\hbar^2 k_z^2}{2m^*}$ та звести задачу до двовимірного рівняння Шредінгера в площині перерізу нанотрубки [6].

Через відсутність радіальної симетрії у шестикутній геометрії, розділення змінних у рівнянні (3.6) ускладнюється. Для математичного розв'язання задачі використано метод апроксимації (метод Бете), де реальний шестикутний переріз замінюється еквівалентним кільцевим шаром з радіусами ρ_0 та ρ_1 [6, 7]. Це дозволяє представити радіальну частину хвильової функції через лінійну комбінацію функцій Бесселя та модифікованих функцій Бесселя [8, 9]:

$$R_m(\rho) = \begin{cases} A_m I_m(\chi_0\rho), & \rho < \rho_0 \\ B_m J_m(k\rho) + C_m N_m(k\rho), & \rho_0 \le \rho \le \rho_1 \\ D_m K_m(\chi_2\rho), & \rho > \rho_1 \end{cases} \tag{3.15}$$

Тут параметри k, χ_0, χ_2 залежать від енергії E та мас носіїв у кожній області [9, 10]. Використання стандартних умов неперервності та збереження густини потоку ймовірності на границях поділу середовищ дає дисперсійне рівняння, розв'язки якого визначають дискретні рівні енергії електрона [10, 11].

Результати розрахунків вказують на те, що гексагональна форма трубки призводить до додаткового зняття виродження рівнів та перерозподілу густини ймовірності, що є критичним для розуміння оптичних властивостей таких наносистем [12, 13].

R E F E R E N C E S

1. Давидов О. С. *Квантова механіка: підручник.* Київ: ВД «Академперіодика», 2012. 708 с.

2. Вакарчук І. О. *Квантова механіка: підручник.* 3-тє вид., доп. Львів: ЛНУ імені Івана Франка, 2007. 848 с.

3. Висоцький В. І. *Квантова механіка та її використання у прикладній фізиці.* Київ: Київський університет, 2008. 367 с.

4. Holovatsky V. A., Gutsul V. I. Electron energy spectrum and wave functions in complicated elliptic quantum wires. *Journal of Optoelectronics and Advanced Materials.* 2007. Vol. 9, No. 5. P. 1437–1441.

5. Harrison P. *Quantum Wells, Wires and Dots: Theoretical and Computational Physics of Semiconductor Nanostructures.* Chichester: John Wiley & Sons, 2016. 616 p.

6. Bastard G. *Wave Mechanics Applied to Semiconductor Heterostructures.* Les Ulis: Les Editions de Physique, 1988. 350 p.

7. Abramowitz M., Stegun I. A. *Handbook of Mathematical Functions.* New York: Dover Publications, 1972. 1046 p.

8. Стрижак П. Є. *Квантова хімія: підручник.* Київ: Києво-Могилянська академія, 2009. 458 с.

9. Whittaker E. T., Watson G. N. *A Course of Modern Analysis.* Cambridge: Cambridge University Press, 1996. 608 p.

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AN EXPONENTIALLY CONVERGENT METHOD FOR A FRACTIONAL–ORDER EVOLUTION EQUATION IN A BANACH SPACE

In a Banach space E, we consider the initial-value problem for an abstract evolution equation within the setting of [1]:

$$\partial_t u + \partial_t^{-\alpha} Au = f(t), \quad t \in (0,\, T],$$
$$u(0) = u_0,$$

where $A : D(A) \to E$ is a strongly positive operator, $f : [0,\, T] \to E$ is a given vector-valued function, $u_0 \in E$ is a given vector, $u : [0,\, T] \to E$ is a sought solution, $\partial_t = \frac{d}{dt}$, and

$$(\partial_t^{-\alpha} u)(t) = \begin{cases} \partial_t \int_0^t \dfrac{(t-s)^\alpha}{\Gamma(1+\alpha)} u(s)\, ds & \text{if} \quad -1 < \alpha \leqslant 0, \\ \int_0^t \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\, ds & \text{if} \;\; 0 < \alpha < 1. \end{cases}$$

In various applications, A is a linear second-order partial differential operator characterized by time-independent coefficients and spatial derivatives.

For convenience, we recall the following definition.

Definition [2]. *A closed linear operator $A : D(A) \to E$ with dense domain $D(A) \subset E$ and resolvent set $\rho(A) \subset \mathbb{C}$ is called strongly positive if there exist constants $\varphi \in (0,\, \pi/2)$, $\gamma > 0$, and $L > 0$ such that*

$$\big\|(zI - A)^{-1}\big\| \leqslant \frac{L}{1+|z|} \;\; \textit{for all } z \in \Sigma,$$

where I denotes the identity operator, and $\Sigma = \big\{z \in \mathbb{C} \,|\, \varphi \leqslant |\arg z| \leqslant \pi\big\} \cup \big\{z \in \mathbb{C} \,|\, |z| \leqslant \gamma\big\} \subset \rho(A)$.

The solution $u(t)$ can be formally expressed in the form:

$$u(t) = U(t)u_0 + \int_0^t U(t-s) f(s)\, ds,$$

where $U(t) = \sum\limits_{n=0}^{\infty} p_n^{(\alpha)}(t^{1+\alpha}) Q^n$ is a solving operator, $Q = A(I+A)^{-1}$ is the Cayley transform of the operator A, and $p_n^{(\alpha)}(t^{1+\alpha})$, $n = 0, 1, \ldots,$ are the Laguerre–Cayley functions given explicitly by the formula:

$$p_n^{(\alpha)}(t^{1+\alpha}) = \sum_{r=0}^{n-1} \frac{(-1)^{r+1} C_{n-1}^r t^{(r+1)(\alpha+1)}}{\Gamma\big(1 + (r+1)(\alpha+1)\big)}, \; n \in \mathbb{N}, \quad p_0^{(\alpha)}(t^{\alpha+1}) \equiv 1.$$

Next, we approximate $u(t)$ as follows:

$$u_N(t) = U_N(t)u_0 + \int_0^t U_N(s) f(t-s)\, ds, \quad U_N(t) = \sum_{n=0}^{N} p_n^{(\alpha)}(t^{1+\alpha}) Q^n. \tag{1}$$

We investigate the accuracy of $u_N(t)$ under specified conditions on the input data u_0 and $f(t)$, assuming that the Laguerre–Cayley functions $p_n^{(\alpha)}(t^{1+\alpha})$ satisfy the inequality

$$\left|p_n^{(\alpha)}(t^{1+\alpha})\right| \leqslant C(t)n^{\gamma}, \tag{2}$$

where $\gamma \in \mathbb{R}$ and $C(t) > 0$ is independent of n. This is supported by extensive numerical evaluations conducted using the computer algebra system Maple.

Let $\nu > 0$ and let $(m_n)_{n=0}^{\infty}$ be a nondecreasing sequence of positive numbers. On the set $C^{\infty}(A) = \bigcap\limits_{n=0}^{\infty} D(A^n)$ of infinitely differentiable vectors of the operator A, we introduce the norm [3]: $\|f\|_{C(A,\,(m_n),\,\nu)} = \sup\limits_{n\in\mathbb{N}_0} \dfrac{\|A^n f\|}{m_n \nu^n}$.

For infinitely smooth u_0 and $f(t)$, we obtain the following result [4].

Theorem. *Let the Laguerre–Cayley functions $p_n^{(\alpha)}(t^{1+\alpha})$ be subject to condition (2) with $\sup\limits_{t\in[0,\,T]} C(t) < \infty$ and $\int_0^t C^2(s)\,ds < \infty$, let $u_0 \in C\left(A,(1),\nu\right)$ with $\nu = \frac{\cos(\varphi)}{L+1}$, and let the right-hand side $f(t)$ satisfy the conditions:*

$$f(t) \in C\left(A,(1),\nu\right), \sup_{t\in[0,\,T]} \|f(t)\|_{C(A,\,(1),\,\nu)} < \infty, \int_0^t \|f(s)\|^2_{C(A,\,(1),\,\nu)} ds < \infty.$$

Then, the Cayley transform method (1) is exponentially convergent, and the following estimate holds:

$$\|u(t) - u_N(t)\| \leqslant M(t)\frac{e^{-\sqrt{N+1}}}{\sqrt{N+1}}\Big\{\|u_0\|_{C(A,\,(1),\,\nu)} + \left[\int_0^t \|f(s)\|^2_{C(A,\,(1),\,\nu)}\,ds\right]^{1/2}\Big\},$$

where $M(t) > 0$ is independent of N.

The case of the finitely smooth input data u_0 and $f(t)$ was studied in [5].

R E F E R E N C E S

1. McLean, W., Thomée, V. *Numerical solution via Laplace transforms of a fractional order evolution equation*, J. Integral Equ. Appl., **22**(1), 57–94 (2010).

2. Gavrilyuk, I. P., Makarov, V. L. *The Cayley transform and the solution of an initial value problem for a first order differential equation with an unbounded operator coefficient in a Hilbert space*, Numer. Funct. Anal. Optim., **15**(5–6), 583–598 (1994).

3. Gorbachuk, V. I., Gorbachuk, M. L. *Boundary Value Problems for Operator-Differential Equations*, Naukova Dumka, Kyiv (1984).

4. Makarov, V. L., Mayko, N. V., Ryabichev, V. L. *Exponentially convergent method for solving a fractional-order evolution equation in a Banach space.* Cybern. Syst. Anal., **62**, 295–304 (2026).

5. Makarov, V. L., Mayko, N. V., Ryabichev, V. L. *A numerical method without accuracy saturation for a fractional-order evolution equation in a Banach space.* J. Appl. Numer. Anal., **3**, 102–112 (2025).

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REGULARITY FOR DOUBLY NONLINEAR EQUATIONS IN THE MIXED REGIME

Let Ω be a domain in $\mathbb{R}^N$, $T > 0$, $\Omega_T := \Omega \times (0, T)$. We study nonnegative (sub) super-solutions to the equation

$$u_t - \operatorname{div}\mathbf{A}(x, t, u, Du^m) = 0, \qquad (x, t) \in \Omega_T. \tag{20}$$

Throughout the paper we suppose that the functions $\mathbf{A} = (A_1, \cdots, A_N) : \Omega_T \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory, i.e. such that $\mathbf{A}(\cdot, \cdot, u, \xi)$ are Lebesgue measurable for all $u \in \mathbb{R}_+$, $\xi \in \mathbb{R}^N$, and $\mathbf{A}(x, t, \cdot, \cdot)$ are continuous for almost all $(x, t) \in \Omega_T$. We also assume that the following structure conditions are satisfied

$$\begin{cases} \mathbf{A}(x, t, u, Du^m) \cdot Du^m \geqslant K_1 \, |Du^m|^p, \\ |\mathbf{A}(x, t, u, Du^m)| \leqslant K_2 \, |Du^m|^{p-1}, \end{cases} \tag{21}$$

where $m > 0$, $p > 1$, and K_1, K_2 are positive constants.

Definition 1. *We say that a function u is a nonnegative, locally bounded, local weak (sub) supersolution to (20) if*

$$0 \leqslant u \in C_{loc}(0, T; L^{1+m}_{loc}(\Omega)), \qquad u^m \in L^p_{loc}(0, T; W^{1,p}_{loc}(\Omega)), \qquad u \in L^\infty_{loc}(\Omega_T),$$

and for any compact set $E \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0, T]$ there holds

$$\int\limits_E u\zeta \, dx \Big|_{t_1}^{t_2} + \int\limits_{t_1}^{t_2} \int\limits_E \{-u\zeta_\tau + \mathbf{A}(x, \tau, u, Du^m) \cdot D\zeta\} \, dx d\tau \leqslant (\geqslant)\, 0, \tag{22}$$

for any testing functions $\zeta \in W^{1,1+\frac{1}{m}}_{loc}(0, T; L^{1+\frac{1}{m}}(E)) \cap L^p_{loc}(0, T; W^{1,p}_0(E))$, $\zeta \geqslant 0$.
A function that is both a weak sub and supersolution to (20) is called a weak solution to (20); in such a case, (1.6) becomes an integral identity.

Remark 1. *In order to simplify the presentation, we have decided to give a definition of solution that already encodes the nonnegativity of u and its local boundedness, besides a certain integrability of the gradient of small powers of u. For these topics, we refer to [1] and references therein.*
Our main result reads as follows. Let the main degeneracy exponent be

$$\lambda = m(p - 1)\,.$$

Theorem 1. [2] *Let u be a nonnegative, local weak solution to (20)-(21) in Ω_T and assume also that one of the following conditions holds*

$$\lambda < 1 \quad \textit{and} \quad p + N(\lambda - 1) > 0, \tag{23}$$

or

$$\lambda > 1 \quad \textit{and} \quad \frac{2N}{N+1} < p < 2, \tag{24}$$

or

$$\lambda > 1 \quad \textit{and} \quad p > 2, \tag{25}$$

then u is locally Hölder continuous in Ω_T.

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R E F E R E N C E S

1. Ciani S., Henriques E., Savchenko M.O., Skrypnik I.I. Parabolic De Giorgi classes with doubly nonlinear, nonstandard growth: local boundedness under exact integrability assumptions, Journal of Differential Equations **462**, 114235 (2026).

2. Ciani S., Henriques E., Savchenko M., Skrypnik I., Y. Yevgenieva Regularity for Doubly Nonlinear Equations in the Mixed Regime, Adv. Calc. Var., to appear.

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CLASSES OF EXTENSIONS OF GRADIENTS ASSOCIATED WITH MONOGENIC FUNCTIONS

We consider classes of piecewise smooth extensions across a smooth curve for gradients of biharmonic functions in the corresponding domains adjacent to the given curve.

Each gradient in every domain is expressed as a pair of some real-valued components of monogenic function in the biharmonic algebra. A theory of biharmonic functions in the biharmonic algebra may be found in [1].

Necessary and sufficient conditions have been found [2] for the existence of an extension across a smooth curve for the gradients of functions for each considered classes. Moreover, the found extension determines the gradient of the biharmonic function in the union of the indicated domains and has certain smooth properties at the curve's points.

R E F E R E N C E S

1. Gryshchuk, S.V., Plaksa, S.A. *Basic Properties of Monogenic Functions in a Biharmonic Plane.* In: M. L. Agranovskii, M. Ben-Artzi, G. Galloway, L. Karp, V. Maz'ya (eds.), "Complex Analysis and Dynamical Systems V". Contemporary Mathematics, **591**, Amer. Math. Soc., Providence, RI, 127–134 (2013).

2. Gryshchuk,S.V. *Piecewise biharmonic extension of gradients and monogenic functions*, J. Math. Sci., **288**(5), 590–603 (2025).

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ПРО ЗАГАЛЬНИЙ РОЗВ'ЯЗОК РІВНЯНЬ РІВНОВАГИ ЛАМЕ ДЛЯ ПРУЖНОЇ СМУГИ З ФУНКЦІОНАЛЬНО-ГРАДІЄНТНОГО МАТЕРІАЛУ

Застосування функціонально-градієнтних матеріалів (ФГМ) у сучасній аерокосмічній галузі, машинобудуванні та триботехніці дає змогу суттєво зменшити концентрацію напружень на межах розділу різних матеріалів та підвищити зносостійкість контактуючих елементів. Аналіз механічних характеристик елементів конструкцій з ФГМ потребує розробки ефективних математичних моделей контактної взаємодії штампів та неоднорідних основ різних геометрій.

Чисельні розв'язки задач гладкого контакту з відривом для пружної смуги, модуль зсуву якої змінюється з товщиною за експоненціальним законом (ФГМ), отримано в роботах S. El- Borgi, R. Abdelmoula та L. Keer [1], I. Comez [2], J. Yan та X. Li [3], G. Adiyaman, A. Birinci, E. Oner та M. Yaylaci [4], I. Comez, S. El-Borgi, V. Kahya та R. Erdol [5].

Метою даної задачі є 1) побудова математичної моделі гладкого контакту між півнескінченним штампом із прямолінійною основою та пружною смугою з функціонально-градієнтного матеріалу, яка має жорстко закріплену нижню грань, 2) аналітична побудова загального розв'язку рівнянь рівноваги в переміщеннях (рівнянь Ламе).

У даній задачі розглядається плоска деформація пружної смуги товщиною h $(-\infty < x < \infty, 0 \leq y \leq h)$. Матеріал смуги є функціонально-градієнтним із експоненціальним законом зміни модуля зсуву по товщині:

$$G(y) = G_0 e^{\gamma y},$$

де $G_0 = G(0)$ — модуль зсуву на верхній грані смуги $(y = 0)$, $G_h = G(h)$ — модуль зсуву на нижній грані смуги $(y = h)$, а γ — параметр градієнтності матеріалу, що визначається співвідношенням

$$\gamma = \frac{1}{h} \ln\left(\frac{G_h}{G_0}\right).$$

Коефіцієнт Пуассона ν вважається сталою величиною вздовж всієї товщини смуги.

У верхню грань смуги $(y = 0)$ вдавлюється півнескінченний жорсткий штамп $(0 < x < \infty)$ із горизонтальною основою під дією рівномірно розподіленого нормального навантаження інтенсивності P, спричиняючи просідання смуги на величину δ. Контакт між штампом та смугою вважається гладким (сили тертя відсутні). Нижня грань смуги $(y = h)$ є жорстко закріпленою.

Сформулюємо крайові умови контактної задачі. На верхній грані смуги $(y = 0)$:

$$\begin{cases} u_y(x,0) = \delta, \quad \tau_{xy}(x,0) = 0, \quad 0 < x < \infty, \\ \sigma_y(x,0) = 0, \quad \tau_{xy}(x,0) = 0, \quad -\infty < x < 0. \end{cases} \tag{26}$$

На нижній грані смуги $(y = h)$:

$$u_x(x,h) = 0, \quad u_y(x,h) = 0, \quad -\infty < x < \infty, \tag{27}$$

тут $u_x(x, y), u_y(x, y)$ — компоненти вектора переміщень; $\sigma_y(x, y), \tau_{xy}(x, y)$ — нормальні та дотичні напруження відповідно.

Отже, сформульовано математичну постановку контактної задачі про взаємодію півнескінченного жорсткого штампа з пружною смугою з ФГМ при плоскій деформації. Виведено рівняння рівноваги Ламе в переміщеннях для середовища з експоненціально змінним за товщиною модулем зсуву $G(y) = G_0 e^{\gamma y}$.

За допомогою інтегрального перетворення Фур'є отримано загальний розв'язок рівнянь рівноваги Ламе. Отримані результати є теоретичною основою для побудови аналітичних і чисельних алгоритмів обчислення розподілу контактних напружень та дослідження впливу параметра градієнтності γ на концентрацію напружень поблизу кутової точки штампа ($x = 0$).

Л І Т Е Р А Т У Р А

1. S. El-Borgi, R. Abdelmoula, L. Keer *A receding contact plane problem between a functionally graded layer and a homogeneous substrate*, Int. J. Solids Struct. **46**(3-4), 658–674 (2006).

2. Comez I. *Contact problem of a functionally graded layer resting on a Winkler foundation*, Acta Mech. **224**(11), 2833–2843 (2013).

3. J. Yan, X. Li *Double receding contact plane problem between a functionally graded layer and an elastic layer*, Eur. J. Mech. and Solids. **53**, 143–150 (2015).

4. G. Adiyaman, A. Birinci, E. Oner, M. Yaylaci *A receding contact problem between a functionally graded layer and two homogeneous quarter planes*, Acta Mech. **227**(6), 1753–1766 (2016).

5. I. Comez, S. El-Borgi, V. Kahya, R. Erdol *Receding contact problem for two-layer functionally graded media indented by a rigid punch*, Acta Mech. **227**(9), 2493–2504 (2016).

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THE USE OF A TWO-DIMENSIONAL KINEMATIC LINEAR KALMAN FILTER WITH ROBUST HUBER CORRECTION FOR FILTERING A HEART RATE SIGNAL

The Kalman filter is one of the most common and at the same time one of the most effective methods of digital signal filtering. It is a powerful mathematical tool for recursively estimating the state of dynamical systems, which is widely used to clean noisy signals [1].

Let's say we constantly receive data from a sensor that somehow characterizes an object. In this case, it is an optical heart rate sensor that is used to obtain a person's pulse waveform, which provides information about the person's heart rate. We know that the sensor is inaccurate (because all sensors have a certain error, which cannot be accurately determined without complex and laborious measurements using reference means). An additional feature of this problem is that we do not have all the data at once, they come in turn. That is, we cannot use classical statistical methods such as averaging, regression, etc.

We roughly know that the pulse should be a periodic (rather — quasi-periodic) function, but we cannot predict anything in advance: neither the shape of this function, nor its period, and even more so — we do not have an analytical expression for it. It cannot even exist, since a person's pulse depends on a large number of factors, and primarily on physical activity at the moment. It turns out that even under such conditions, the Kalman filter can be used. However, if a one-dimensional Kalman filter is used for this problem (the signal in this case is one-dimensional), then in its practical implementation on any microcontroller, the effect of "over-smoothing" will inevitably appear, as a result of which the output signal will lose small details. To prevent this from happening, it is necessary to use a two-dimensional (2D) kinematic linear Kalman filter with robust Huber correction [2].

We model the signal not as a static number, but as a physical object that moves in time and has a value and a rate of change. At each discretization step k, the state of the system is described by a vector 2×1: $\mathbf{x}_k = \begin{bmatrix} x_k \\ \dot{x}_k \end{bmatrix}$ where x_k is an estimate of the true value of the PPG signal and $\dot{x}_k$ is the rate of change of the signal (first derivative, $\frac{dx}{dt}$).

We assume that for a short time interval Δt (discretization period, for example, $\Delta t = 0.02$ s for 50 Hz) the process is described by the equations of uniform motion: $x_k = x_{k-1} + \dot{x}_{k-1} \cdot \Delta t + w_{x,k-1}$; $\quad \dot{x}_k = \dot{x}_{k-1} + w_{\dot{x},k-1}$. In matrix form:

$$\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k-1} + \mathbf{w}_{k-1},$$

where F is the state transition matrix:

$$F = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix},$$

and $\mathbf{w}_{k-1}$ is the process noise vector that takes into account unpredictable changes in speed (acceleration of the heart rate).

The uncertainty of the model is described by the covariance matrix Q of size 2×2:

$$Q = \sigma_a^2 \begin{bmatrix} \frac{\Delta t^4}{4} & \frac{\Delta t^3}{2} \\ \frac{\Delta t^3}{2} & \Delta t^2 \end{bmatrix},$$

where σ_a^2 is the parametric variance of the signal "acceleration" (the tuning parameter of the filter flexibility).

Before updating the state, we check whether the new measurement z_k is not an outlier. To do this, first calculate the value of the residual: $y_k = z_k - H\hat{\mathbf{x}}_k^-$, where z_k is raw sensor data, its variance is $S_k = HP_k^- H^T + R = P_{k,(1,1)}^- + R$, and the standardized error is

$$e_k = \frac{y_k}{\sqrt{S_k}}.$$

Based on the last parameter, the concept of Huber weight (w_k) is introduced [1]: for a given threshold $c \in [1.3; 1.8]$

$$w_k = \begin{cases} 1, & \text{if } |e_k| \leq c \\ \frac{c}{|e_k|}, & \text{if } |e_k| > c \end{cases}$$

Then adapted measurement noise is $\tilde{R}_k = \frac{R}{w_k}$ and adjusted innovation variance is $\tilde{S}_k = P_{k,(1,1)}^- + \tilde{R}_k$.

This processing step computes a matrix of Kalman gains $K_k = P_k^- H^T \tilde{S}_k^{-1}$, which determines how much the filter will adjust its prediction based on the new measurement. For our 1D measurement, this simplifies to a vector:

$$K_k = \frac{1}{\tilde{S}_k} \begin{bmatrix} P_{k,(1,1)}^- \\ P_{k,(2,1)}^- \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}.$$

At the final stage, we adjust our knowledge of the system and obtain the final (a posteriori) estimates, i.e. the final filtered signal):

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + K_k y_k.$$

The last expression is calculated component-wise as follows:

$$\hat{x}_k = \hat{x}_k^- + K_1 \cdot y_k; \qquad \hat{\dot{x}}_k = \hat{\dot{x}}_k^- + K_2 \cdot y_k.$$

The Kalman filter improved in this way provides good smoothing of the output signal, but at the same time, small details that are important for medical diagnostics based on the heart rate signal are also quite noticeable.

R E F E R E N C E S

1. Haykin S. *Adaptive Filter Theory. 5th ed.* Boston : Pearson. 912 p. (2014).

2. Becker A. *Kalman Filter from the Ground Up* KalmanFilter.NET. [Online]. Available: https://www.kalmanfilter.net/. [Accessed: Aug. 13, 2026].

Секція 4

Математичний та комплексний аналіз / Mathematical and Complex Analysis

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ω_n-SYMMETRIC POLYNOMIALS ON BANACH SPACES

Let $\omega_n = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ be the group of nth roots of unity so that $\alpha_0 = 1$, and $\alpha_1^j = \alpha_j = e^{i2\pi j/n}$, $0 \le j < n$. We construct the set of indexes $\mathbb{N}_{\omega_n}$ as a matrix n rows with entries $m\alpha_j$ for $0 \le j \le n-1$ and $m \in \mathbb{N}$. For every $1 \le p < \infty$, we denote by $\ell_p(\mathbb{N}_{\omega_n})$ the Banach space of all absolutely summable power p complex sequences indexed by numbers in $\mathbb{N}_{\omega_n}$. Any vector x in $\ell_p(\mathbb{N}_{\omega_n})$ can be represented as

$$x = \big(x_{m\alpha_j}\big)_{m\in\mathbb{N},\ 0\le j<n} = \begin{pmatrix} x_{\alpha_0} \\ x_{\alpha_1} \\ \vdots \\ x_{\alpha_{n-1}} \end{pmatrix} = \begin{pmatrix} x_{1\alpha_0}, & x_{2\alpha_0}, & \cdots, & x_{m\alpha_0}, & \cdots \\ x_{1\alpha_1}, & x_{2\alpha_1}, & \cdots & x_{m\alpha_1}, & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{1\alpha_{n-1}}, & x_{2\alpha_{n-1}}, & \ldots, & x_{m\alpha_{n-1}}, & \ldots \end{pmatrix},$$

where $x_{\alpha_j} = (x_{1\alpha_j}, x_{2\alpha_j}, \ldots, x_{m\alpha_j}, \ldots)$ is in ℓ_p for all $j \in \{0, 1, 2, \ldots, n-1\}$, and

$$\|x\| = \Big(\sum_{i=1}^{\infty}\sum_{j=0}^{n-1} |x_{i\alpha_j}|^p\Big)^{\frac{1}{p}}.$$

A polynomial P on $\ell_p(\mathbb{N}_{\omega_n})$ is called *ω_n-symmetric* if it is a finite algebraic combination of polynomials $\{\Omega_k^{(n)}\}_{k=1}^{\infty}$, where

$$\Omega_k^{(n)}(x) = \sum_{j=0}^{n-1} \alpha_j \sum_{i=1}^{\infty} x_{i\alpha_j}^k, \quad k \in \mathbb{N}, \quad k \ge \lceil p \rceil.$$

The algebra of all ω_n-symmetric polynomials on $\ell_p\big(\mathbb{N}_{\omega_n}\big)$ is denoted by $\mathcal{P}_s\big(\ell_p(\mathbb{N}_{\omega_n})\big)$. An ω_1-symmetric polynomial is called *symmetric*.

The concept of ω_n-symmetric polynomials on ℓ_1 was introduced in [1]. In this talk we consider more general case of ℓ_p and discuss some new properties of ω_n-symmetric polynomials.

For vectors x, $y \in \ell_p$ we denote $x \bullet y = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots)$.

Theorem 1. *Let $P \in \mathcal{P}_s\big(\ell_p(\mathbb{N}_{\omega_n})\big)$ and $a \in \ell_p$. Then*

$$P\begin{pmatrix} x_{\alpha_0} \bullet a \\ x_{\alpha_1} \bullet a \\ \vdots \\ x_{\alpha_{n-1}} \bullet a \end{pmatrix} = P\begin{pmatrix} x_{\alpha_0} \\ x_{\alpha_1} \\ \vdots \\ x_{\alpha_{n-1}} \end{pmatrix} \quad \textit{for every} \quad x = \begin{pmatrix} x_{\alpha_0} \\ x_{\alpha_1} \\ \vdots \\ x_{\alpha_{n-1}} \end{pmatrix} \in \ell_p(\mathbb{N}_{\omega_n}).$$

For a given $z \in \ell_p$ we define $x_z \in \ell_p(\mathbb{N}_{\omega_n})$ as

$$x_z = \begin{pmatrix} \alpha_0 z \\ \alpha_1 z \\ \vdots \\ \alpha_{n-1} z \end{pmatrix}.$$

Theorem 2. *For every $k \geq \lceil p \rceil$ and integer n, the map $z \mapsto \Omega_k^{(n)}(x_z)$ is a symmetric polynomial on ℓ_p.*

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R E F E R E N C E S

1. Chopiuk, Y., Zagorodnyuk, A. *Symmetric functions and rings of multinumbers associated with finite groups*, Symmetry **17**, 33 (2025).

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TOTALLY POSITIVE SEQUENCES GENERATED BY REAL POLYNOMIALS

Definition. A sequence of nonnegative numbers $(a_k)_{k=0}^{\infty}$ is called totally positive, if all minors of the infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are non-negative.

Concept of total positivity found numerous applications and was studied from many different sides. It has applications in distribution of zeros of polynomials and entire functions, Pólya frequency sequences, unimodality and log-concavity, stochastic processes and approximation theory, mechanical systems, planar networks, combinatorics and representation theory.

Let $P, P(0) > 0$, be a real polynomial of degree m. It is easy to check that $\sum_{k=0}^{\infty} P(k)x^k = \frac{Q(x)}{(1-x)^{m+1}}$, where Q is a real polynomial of degree at most m.

We will discuss the following problem: for which P the sequence $(P(k))_{k=0}^{\infty}$ is totally positive? Due to the famous theorem by Aissen, Schoenberg, Whitney and Edrei it happens if and only if all the zeros of Q are real and non-positive.

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GROWTH ESTIMATES FOR ANALYTIC FUNCTIONS IN THE UNIT POLYDISC WITH BOUNDED L-INDEX IN DIRECTION

We will use notations and definitions from [1]. Let $\mathbf{0} = (0, \dots, 0)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{D}^n = \{z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n\colon |z_j| < 1, j \in \{1, 2, \dots, n\}\}$, $L\colon\ \mathbb{D}^n \to \mathbb{R}_+$ be a continuous function such that for all $z = (z_1, z_2, \dots, z_n) \in \mathbb{D}^n$

$$L(z) > \beta \max_{1\le j\le n} \frac{|b_j|}{1-|z_j|}, \quad \beta = \text{const} > 1. \tag{28}$$

An analytic function $F\colon \mathbb{D}^n \to \mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{D}^n$ one has $\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)}\colon\ 0 \le k \le m_0\right\}$, where $\partial_{\mathbf{b}}^0 F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j$, $\partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}}(\partial_{\mathbf{b}}^{k-1} F(z))$, $k \ge 2$.

Theorem 1. *Let $L\colon \mathbb{D}^n \to \mathbb{R}_+$ be a continuous function satisfying (28) and for every $z^0 \in \mathbb{D}^n$ and every $\theta \in [0, 2\pi]$ the function $u(r, z^0, \theta)$ be a continuously differentiable function of real variable $r \in [0, R)$, where $R = R(z^0, \theta) = \min\{R \in \mathbb{R}_+\colon\ |z_j^0 + Re^{i\theta} b_j| = 1,\ j \in \{1, \dots, n\}\}$. If an analytic function $F\colon \mathbb{D}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b}$ then $\forall z^0 \in \mathbb{D}^n$, $\forall \theta \in [0, 2\pi]$, $\forall r \in [0, R)$ $\forall p \in \mathbb{Z}_+$ one has*

$$\ln \max_{|t|=r} \left(\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p! L^p(z^0+t\mathbf{b})}\right) \le \ln\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0)|}{k! L^k(z^0)}\colon 0\le k\le N\right\} +$$
$$+ \max_{\theta\in[0,2\pi]} \int_0^r \left\{(N+1)L(z^0+te^{i\theta}\mathbf{b}) + N\frac{(-u_t'(z^0,\theta,t))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right\} dt$$

If, in addition, for every $z^0 \in \mathbb{D}^n$, $\theta \in [0, 2\pi]$ one has $(-u_r'(z^0, \theta, r))^+ / (L^2(z^0+re^{i\theta}\mathbf{b})) \to 0$ as $\max\limits_{1\le j\le n} |z_j^0 + re^{i\theta} b_j| \to 1$ then for every $z^0 \in \mathbb{D}^n$ and $\theta \in [0, 2\pi]$

$$\varlimsup_{\max_{1\le j\le n} |z_j^0+re^{i\theta}b_j|\to 1} \frac{\ln |F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt} \le N_{\mathbf{b}}(F, L, \mathbb{D}^n) + 1.$$

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R E F E R E N C E S

1. Bandura, A.I., Batsala, Ya.V. *Growth estimates for analytic functions in the unit polydisc with bounded L-index in direction*, Mat. Stud. **65**(2), 148-156 (2026).

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LAPLACE-STIELTHJES TYPE INTEGRALS: THE BOREL RELATION AND THE h-MEASURE OF AN EXCEPTIONAL SET

By $\mathcal{I}(\nu)$ denote the class of functions $F\colon \mathbb{R}\to\mathbb{R}_+$ represented by integrals of the form

$$F(x)=\int_{\mathbb{R}_+} a(t)e^{xt}\nu(dt).$$

For $F\in\mathcal{I}(\nu)$ and $x\in\mathbb{R}$ let us denote

$$\mu_*(x,F)=\operatorname{ess\,sup}\{f(u)e^{xu}\colon u\in S\}.$$

We denote by $\mathcal{L}$ the class of positive continuous functions $\Phi\colon \mathbb{R}_+:=[0,+\infty)\to\mathbb{R}_+$ such that $\Phi(x)\nearrow+\infty$ $(0\le x\to+\infty)$, and by $\mathcal{L}^+$ the class of positive continuous differentiable functions $h(x)\colon[0,+\infty)\to[0,+\infty)$ such that $\int_0^{+\infty}dh(x)=+\infty$ and $1\le h'(x)\nearrow+\infty$ $(0\le x\nearrow+\infty)$. Denote by $\mathcal{L}_1$ the class of function $\psi\in\mathcal{L}$ such that $\int_{x_0}^{+\infty}dt/\psi(t)<+\infty$, $\psi(t)\ge t$ for $t\ge t_0>0$, and by $\mathcal{L}_2$ the class of the functions $\psi\in\mathcal{L}_1$ such that the inverse function ψ^{-1} to function ψ satisfies the condition $\psi^{-1}(uv)\le u\psi^{-1}(v)$ $(u,v\ge1)$. Let $\Phi\in\mathcal{L}$. Let us introduce the following class of Dirichlet series

$$\mathcal{D}(\lambda,\Phi)=\{F\in\mathcal{D}(\lambda)\colon\ \ln\mu(x,F)\ge x\Phi(x)\ (x\ge x_0)\}.$$

For a Lebesgue measurable set $E\subset[0,+\infty)$ we call

$$m_hE\equiv\int_E dh(x)$$

its h-measure, at $h(x)\equiv x$ the h-measure of a set E is its Lebesgue measure.

In the article [6] the following theorem was proved.

Theorem 2. [6] *Let $h\in\mathcal{L}^+$, $\Phi\in\mathcal{L}$. If a function $F\in\mathcal{D}(\lambda,\Phi)$ and a number $b>0$ such that*

$$\sum_{n=1}^{+\infty}\frac{h'(p(\lambda_n)+b)}{n\lambda_n}<+\infty,$$

then asymptotic relation $\ln M(x,F)\sim\ln\mu(x,F)$ holds as $x\to+\infty$ outside some set E of finite h-measure ($m_hE<+\infty$), where the function φ is an inverse function to the function Φ.

In this article we will prove a similar statement for the functions from the class $\mathcal{I}(\nu)$. The proof method differs from the classical Wiman-Valiron type method was used in the article [6].

We say that $F\in\mathcal{I}(\nu,\Phi)$ if $F\in\mathcal{I}(\nu)$ and

$$\ln F(x)\ge x\Phi(x)\quad(x\ge x_0).$$

We prove the following theorem.

Theorem 3. *Let $h \in \mathcal{L}^+$, $\Phi \in \mathcal{L}$, $\Phi_0(x) = x\Phi(x)$. If a function $F \in \mathcal{I}(\nu, \Phi)$ and there exists a function $\psi \in \mathcal{L}_2$ that the condition*

$$h'(\varphi_0(t)) \ln \nu_0(t) = o\Big(\psi^{-1}(t)\Big) \quad (t \to +\infty)$$

satistfy with $\nu_0(t) = \nu([0, t])$, then asymptotic relation $\ln F(x) = (1 + o(1)) \ln \mu_*(x, F)$ holds as $x \to +\infty$ outside some set E of finite h-measure ($m_h E < +\infty$), where the function φ_0 is an inverse function to the function Φ_0.

Remark 1. *It is easy to verify that condition $\int_{x_0}^{+\infty} dx/\psi(x)$ is equivalent to condition $\int_{t_0}^{+\infty} t^{-2}\psi^{-1}(t)dt < +\infty$. On the other hand, for given non-decreasing function $\alpha(t)$ the condition $\int_{t_0}^{+\infty} t^{-2}\alpha(t)dt < +\infty$ is equivalent to the condition that there exists a function $\psi \in \mathcal{L}_1$ such that $\alpha(t) = o(\psi^{-1}(t))$ $(t \to +\infty)$.*

The authors do not know whether it is possible to choose the function ψ so that the condition of belonging to the class $\mathcal{L}_2$ is fulfilled.

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R E F E R E N C E S

Skaskiv O.B. *Behavior of the maximal term of a Dirichlet series that defines an entire function*, Math. Notes V.37, №1. P. 686–693 (1985).

Pólya G., Szegő G. Aufgaben und Lehrsätze aus der Analysis. Berlin: Springer, V.2, (1925).

Wiman A. Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Function und dem grössten Gliebe der zugehörigen Taylorischen Reiche Acta Math. V.37. P. 305–326 (1914).

Valiron G. *Sur les fonctions entieres d'ordre fini et d'ordre null et en particulier les fonctions a correspondance reguliere* Ann. Fac. Sci. Toulouse Math. (6). V.5. P. 117–257 (1914).

Skaskiv O.B., Bandura A.I. Asymptotic estimates of positive integrals and entire functions. Lviv–Ivano-Frankivsk, LNU-INFTUNG, Goliney, 108 p. (2015) (in Ukrainian)

Kuryliak A.O., Salo T.M. *Borel type asymptotic relation for entire Dirichlet series and h-measure of an exceptional sets* Mat. Stud. V.65, no.1. P. 15–21 (2026).

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APPROXIMATION OF FUNCTION BY ZYGMUND METHOD AND THE BEST APPROXIMATION IN WEIGHTED ORLICZ TYPE SPACES WITH VARIABLE EXPONENT

We consider linear methods of summation of Fourier series in weighted Orlicz type spaces $\mathcal{S}_{\mathbf{p},\mu}$ with variable exponent. The spaces $\mathcal{S}_{\mathbf{p},\mu}$ are defined in the following way. Let $\mathbf{p}=\{p_k\}_{k=-\infty}^{\infty}$ be a sequence of positive numbers such that

$$1\le p_k\le K,\quad k=0,\pm1,\pm2,\ldots,\ (k\in\mathbb{Z}),$$

where K is a positive number, and $\mu=\{\mu_k\}_{k=-\infty}^{\infty}$ be a sequence of nonnegative numbers. Let $\mathcal{S}_{\mathbf{p},\mu}$ be the space of all 2π-periodic Lebesgue summable functions f ($f\in L_1$) such that the following quantity (which is also called the Luxemburg norm of f) is finite:

$$\|f\|_{\mathbf{p},\mu}:=\inf\left\{a>0:\ \sum_{k\in\mathbb{Z}}\mu_k|\widehat{f}(k)/a|^{p_k}\le 1\right\}.$$

where $\widehat{f}(k):=(2\pi)^{-1}\int_0^{2\pi}f(t)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t$, $k\in\mathbb{Z}$, are the Fourier coefficients of f. The spaces $\mathcal{S}_{\mathbf{p},\mu}$ defined in this way are the Banach spaces. They were considered in [1].

Let $\mathcal{T}_n$, $n=0,1,\ldots$, be the set of all trigonometric polynomials $\tau_n(x):=\sum_{|k|\le n}c_k\mathrm{e}^{\mathrm{i}kx}$ of the order n, where c_k are arbitrary complex numbers. Let also

$$E_n(f)_{\mathbf{p},\mu}:=\inf_{\tau_{n-1}\in\mathcal{T}_n}\|f-\tau_n\|_{\mathbf{p},\mu}$$

denotes the best approximation of a function $f\in\mathcal{S}_{\mathbf{p},\mu}$ by the trigonometric polynomials $\tau_n\in\mathcal{T}_n$ in $\mathcal{S}_{\mathbf{p},\mu}$.

Further, for any function $f\in L_1$ with the Fourier series of the form

$$S[f](x)=\sum_{k\in\mathbb{Z}}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx},$$

consider for $n=0,1,\ldots,$ the following linear transformations:

$$S_n(f)(x):=\sum_{k=-n}^{n}\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx},\qquad Z_n^{(s)}(f)(x):=\sum_{k=-n}^{n}\left(1-\left(\frac{|k|}{n+1}\right)^s\right)\widehat{f}(k)\mathrm{e}^{\mathrm{i}kx},\ s>0.$$

The expressions $S_n(f)$ and $Z_n^{(s)}(f)$ are called the partial Fourier sum and the Zygmund sum, respectively.

Let $\psi=\{\psi(k)\}_{k\in\mathbb{Z}}$ be a numerical sequence whose members are not all zero and

$$\mathcal{Z}(\psi):=\{k\in\mathbb{Z}:\psi(k)=0\}.$$

In what follows, assume that the number of elements of the set $\mathcal{Z}(\psi)$ is finite.

If for a function $f \in L_1$, there exists a function $g \in L_1$ with the Fourier series of the form

$$S[g](x) = \sum_{k \in \mathbb{Z} \setminus \mathcal{Z}(\psi)} \widehat{f}(k) e^{\mathrm{i}kx} / \psi(k),$$

then we say that for the function f, there exists ψ-derivative g, for which we use the notation $g = f^{\psi}$. We consider ψ-derivatives defined by the sequences of the following form: $\psi(k) = |k|^{-s}$, $k \in \mathbb{Z}$, $s > 0$, and use in this case the notation $f^{(s)}$.

This definition of ψ-derivative is adapted to the needs of the our research and it is not fundamentally different from the established concept of ψ-derivative of A.I. Stepanets [2].

The modulus of smoothness of $f \in \mathcal{S}_{\mathbf{p},\mu}$ of the index $\alpha > 0$ is defined by

$$\omega_\alpha(f,\delta)_{\mathbf{p},\mu} := \sup_{|h|\le\delta} \|\Delta_h^\alpha f\|_{\mathbf{p},\mu} = \sup_{|h|\le\delta} \Big\| \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x - jh) \Big\|_{\mathbf{p},\mu},$$

where $\delta > 0$, $\binom{\alpha}{0} := 1$, $\binom{\alpha}{j} = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-j+1)/j!$, $j \in \mathbb{N}$.

Let ω be a function defined on the interval $[0,1]$. For $\alpha > 0$, we set

$$\mathcal{S}_{\mathbf{p},\mu} H_\omega^\alpha := \left\{ f \in \mathcal{S}_{\mathbf{p},\mu} : \quad \omega_\alpha(f,\delta)_{\mathbf{p},\mu} = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+ \right\}.$$

Let $\mathfrak{B}$ and $\mathfrak{B}_s$, $s \in \mathbb{N}$, denote the sets of all continuous strictly increasing functions $\omega(t)$, $t \in [0,1]$, with $\omega(0) = 0$ satisfying the following conditions, respectively:

$$\sum_{v=n+1}^{\infty} \frac{\omega(1/v)}{v} = \mathcal{O}[\omega(n^{-1})], \quad \sum_{v=1}^{n} v^{s-1}\,\omega(v^{-1}) = \mathcal{O}[n^s \omega(n^{-1})], \quad n \to \infty.$$

Theorem 1. *Assume that $f \in L_1$, $s > 0$ and $\omega \in \mathfrak{B}$. The following statements are equivalent:*

1) $\|S_n(f^{(s)})\|_{\mathbf{p},\mu} = \mathcal{O}(n^s \omega(n^{-1})), \quad n \to \infty;$

2) $\Big\| f - Z_n^{(s)}(f) \Big\|_{\mathbf{p},\mu} = \mathcal{O}(\omega(n^{-1})), \quad n \to \infty;$

3) $f \in \mathcal{S}_{\mathbf{p},\mu} H_\omega^s$.

Furthermore, if one of the statements 1), 2) or 3) is satisfied, then

4) $E_n(f)_{\mathbf{p},\mu} = \mathcal{O}(\omega(n^{-1})), \quad n \to \infty;$

If $\omega \in \mathfrak{B} \cap \mathfrak{B}_s$, then the statements 1)–4) *are equivalent.*

R E F E R E N C E S

1. Abdullayev F., Chaichenko S., Imash kyzy M., Shidlich A. *Direct and inverse approximation theorems in the weighted Orlicz-type spaces with a variable exponent*, Turkish Journal of Mathematics **44**, (1), 284-299 (2020).

2. Stepanets A. I. *Methods of approximation theory*, Leiden, VSP, (2005).

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ON THE UNIFORM BOUNDARY SEPARATION FOR MAPPINGS IN VARIABLE DOMAINS

Let $\mathbb{R}^n$, $n \geqslant 2$, be the n-dimensional Euclidean space. The boundary ∂A and the closure $\overline{A}$ of a set $A \subset \mathbb{R}^n$ are considered with respect to the extended space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, and $h(x, y)$ stands for the standard chordal metric. For a given set $E \subset \overline{\mathbb{R}^n}$, its chordal diameter is defined as $h(E) = \sup_{x,y\in E} h(x, y)$.

A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called admissible for a family Γ of paths γ in $\mathbb{R}^n$, if the relation $\int_\gamma \rho(x)|dx| \geqslant 1$ holds for all locally rectifiable paths $\gamma \in \Gamma$. In this case, we write $\rho \in adm\,\Gamma$. For $p \geqslant 1$, the p-modulus of Γ is defined by the equality

$$M_p(\Gamma) = \inf_{\rho\in adm\,\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)dm(x), \tag{29}$$

where $dm(x)$ is the element of the Lebesgue measure in $\mathbb{R}^n$.

For $x_0 \in \mathbb{R}^n$ and $0 < r_1 < r_2 < \infty$, denote $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$ and $A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$. Given a domain $D \subset \mathbb{R}^n$ and a measurable function $Q : \mathbb{R}^n \to [0, \infty]$, a mapping $f : D \to \mathbb{R}^n$ is called a ring Q-mapping at a point $x_0 \in D$ with respect to the p-modulus, if the condition

$$M_p(f(\Gamma(S(x_0, r_1), S(x_0, r_2), D))) \leqslant \int\limits_{A(x_0,r_1,r_2)\cap D} Q(x) \cdot \eta^p(|x - x_0|)dm(x) \tag{30}$$

holds for any $0 < r_1 < r_2 < d(x_0, \partial D)$ and every measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that $\int_{r_1}^{r_2} \eta(r)dr \geqslant 1$. All other basic definitions, including functions of finite mean oscillation (FMO), can be found in [1].

For $x_0 \in \mathbb{R}^n$, denote

$$q_{x_0}(r) = \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{|x-x_0|=r} Q(x)\, dS,$$

where dS is the surface area element and ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. We say that Q satisfies the integral divergence condition at $x_0 \in \mathbb{R}^n$ if

$$\int\limits_0^{\delta_0} \frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)} = \infty \tag{31}$$

for some $\delta_0 > 0$.

We study the problem of the uniform boundary separation for images of compact sets under spatial mappings in variable domains. Let $D_0 \subset \mathbb{R}^n$ be a domain, and let $\mathfrak{D} = \{D_m\}_{m=1}^{\infty}$ be a sequence of domains the kernel of which (in the sense of Carathéodory) is D_0.

Theorem 1. *Let A be a non-degenerate continuum in D_0 such that $A \subset \bigcap_{m=1}^{\infty} D_m$. Let $f_m : D_m \to \mathbb{R}^n$ be a sequence of open, discrete, and closed mappings that are ring Q-mappings at any point $x_0 \in D_0$ with respect to the n-modulus. Assume that the following conditions are satisfied:*

1) the majorant Q has a finite mean oscillation in A (or satisfies the corresponding integral divergence condition (3) for every $x_0 \in A$);

2) there exists $\delta > 0$ such that $h(f_m(A)) \geqslant \delta$ for all $m \geqslant 1$;

3) there is $r > 0$ such that $h(E) \geqslant r$ whenever E is any component of $\partial f_m(D_m)$.

Then there exists a positive constant $\delta_1 > 0$ such that the separation condition

$$h(f_m(A), \partial f_m(D_m)) \geqslant \delta_1 \tag{32}$$

holds for every $m \geqslant 1$.

Furthermore, if A_0 is some another compactum in D_0, then there are $\delta_2 > 0$ and $M_0 \in \mathbb{N}$ such that $A_0 \subset D_m$ for all $m \geqslant M_0$ and the uniform separation condition

$$h(f_m(A_0), \partial f_m(D_m)) \geqslant \delta_2 \tag{33}$$

holds for every $m \geqslant M_0$.

The detailed proofs of the formulated results established in [2].

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R E F E R E N C E S

1. Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. *Moduli in Modern Mapping Theory*, Springer Science + Business Media, LLC, New York (2009).

2. Ilkevych, N., Romash, D., Sevost'yanov, E. *On distortion estimates under mappings of families of domains*, Preprint, https://arxiv.org/abs/2602.14013 (accepted for publication in Filomat) (2026).

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FOURIER QUASICRYSTALS: THEOREMS AND PROBLEMS

The Fourier transform
$$\hat{\varphi}(x) = \int_{\mathbb{R}} \varphi(t)e^{-2\pi itx}dt,$$
is well-defined for all φ belonging to Schwartz' class $\mathcal{S}$ of C^∞ with the properties
$$t^m\varphi^{(n)}(t) \to 0 \qquad \forall m,n \in \mathbb{N}\cup\{0\}, \qquad |t|\to\infty,$$
For a tempered distribution $f \in \mathcal{S}$ the Fourier transform is defined by the formula
$$\hat{f}(\varphi) := f(\hat{\varphi}), \qquad \forall\varphi\in\mathcal{S}.$$
Here we consider only the very rare but extremely important case where f and $\hat{f}$ are measures of the form
$$f = \mu = \sum_{\lambda\in\Lambda} a_\lambda\delta_\lambda, \qquad \hat{f} = \hat{\mu} = \sum_{\gamma\in\Gamma} b_\gamma\delta_\gamma.$$
The above definition is equivalent to the Poisson formula
$$\sum_{\gamma\in\Gamma} b_\gamma\varphi(\gamma) = \sum_{\lambda\in\Lambda} a_\lambda\hat{\varphi}(\lambda), \qquad \forall\varphi\in\mathcal{S}.$$

If Λ and Γ are locally finite (intersections with any interval are finite), then μ is a crystalline measure, and if the variations $|\mu|$, $|\hat{\mu}|$ belong to $\mathcal{S}^*$, then μ is the Fourier quasicrystal. In 2023, I constructed the first example of a crystalline measure, which is not the Fourier quasicrystal.

A measure μ is called translation bounded if $\sup_{x\in\mathbb{R}}|\mu|(x,x+1)<\infty$.

Theorem 1. (S.Favorov 2024) *If a measure $\mu\in\mathcal{S}^*$ non-positive and $\hat{\mu}$ is a measure then μ is translation bounded.*

Theorem 2. *A measure $|\mu|\in\mathcal{S}^*$ if and only if $M_\mu(r) = |\mu|(-r,r)$ grows no more than polynomially.*

Theorem 3. (S. Favorov 2023) *If $\mu\in\mathcal{S}^*$ as above and*
$$\exists\ h>0 \quad \forall\lambda,\lambda'\in\Lambda \qquad |\lambda-\lambda'|\ge c\min\{1,\,|\lambda|^{-h}\}.$$
then $|\mu|\in\mathcal{S}^*$

Theorem 4. (P. Boivalenkov, S. Favorov 2025) *Let a measure $\mu\in\mathcal{S}^*$ be translation bounded. Then*
a) the measure $\nu=\sum_{\gamma\in\Gamma}|b_\gamma|^2\delta_\gamma$, where $\Gamma = supp\hat{\mu}$, is also translation bounded,
b) if restrictions of Γ to any interval of length η are linearly independent over $\mathbb{Z}$, then the measure $\hat{\mu}$ is also translation bounded.

Problem. Show that we cannot replace the power 2 in the definition of the measure ν with any smaller one.

In 2020 A. Olevsky and A. Ulanovsky proved 1-1 correspondence between Fourier quasicrystals with unit masses and zeros of real-rooted exponential polynomials

$$P(z) = \sum_{1\le j\le N} q_j e^{2\pi i\omega_j z}, \qquad q_j \in \mathbb{C}, \quad \omega_j \in \mathbb{R}.$$

In 2025, F. Goncalves discovered a one-to-one correspondence between Fourier quasicrystals and a special class of meromorphic functions whose poles coincide with the supports of Fourier quasicrystals, and whose residues coincide with the masses of Fourier quasicrystals at these points.

Generalizations to measures in horizontal strips of a finite width. For $\varphi \in \mathcal{S}$ with compact support consider the extending of $\hat\varphi$ to the entire function on $\mathbb{C}$

$$\hat\varphi^c(z) = \int_{\mathbb{R}} \varphi(t) e^{-2\pi i z t} dt.$$

For a measure μ in a strip $S_H = \{z = x + iy : x \in \mathbb{R},\ |y| \le H\}$ under assumption

$$M_\mu(r) := |\mu|\{z : |x| \le r,\ |y| \le H\} = O(r^N), \qquad \text{for some } N < \infty,$$

set

$$\int \varphi(t)\hat\mu^c(dt) = \int \hat\varphi^c(z`)\mu(dz)$$

In 2025, we proved an analogue of A.Olevskii-A.Ulanovsky's result for measures in a strip and its generalization to the class of Dirichlet series with a bounded spectrum Ω

$$Q(z) = \sum_{\omega\in\Omega} q_\omega e^{i\omega z}, \qquad \sup\Omega \in \Omega, \quad \inf\Omega \in \Omega,$$

We will say that the measure μ on S_H is translation bounded if

$$\sup_{x'\in\mathbb{R}} |\mu|\{z : x' \le x \le x' + 1,\ |y| \le H\} < \infty.$$

Theorem 1*. (S. Favorov, O. Deger 2026) *If a measure $\mu \in \mathcal{S}^*$ is non-negative then it is translation bounded.*

Theorem 4*. (S. Favorov, O. Deger 2026)

$$M_\nu(r) \le C_1 e^{4\pi H r}, \qquad M_{\hat\mu^c}(r) \le C_2 e^{2\pi H r}.$$

It is impossible to remove the constant H from here.

Problems. Generalize other theorems about measures on $\mathbb{R}$, in particular Goncalves' result, Theorem 3, to the case of measures on strips.

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ARGUMENTS OF THE TAYLOR COEFFICIENTS AND GROWTH OF AN ENTIRE FUNCTIONS

We denote by $M_f(r) = \max\{|f(z)|\colon |z| \leq r\}$ maximum modulus of the function

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n = \sum_{n=0}^{+\infty} |a_n| e^{i\theta_n} z^n, \theta_n = \operatorname{agr} a_n, \tag{34}$$

and by

$$g(z) = \sum_{n=0}^{+\infty} |a_n| z^n, \quad \mathfrak{M}_f(r) = M_g(r) = \sum_{n=0}^{+\infty} |a_n| r^n$$

the majorant of the series (34).

The equation how essentially do the arguments of coefficients influence on the growth of its maximum modulus may be reformulated to the following: how quickly may $\mathfrak{M}_f(r)$ grow with respect to $M_f(r)$?

Remark that $M_f(r) \leq \mathfrak{M}_f(r)$ $(r \geq 0)$. But on the other hand, for every nonconstant entire function $f(z)$ there exist a set $E \subset [1; +\infty)$ of finite logarithmic measure $(\int_E r^{-1} dr < +\infty)$ such that

$$0 \leq \varlimsup_{\substack{r\to+\infty \\ r\notin E}} \frac{\ln \mathfrak{M}_f(r) - \ln M_f(r)}{\ln\ln M_f(r)} \leq \frac{1}{2}. \tag{35}$$

Consider random entire functions

$$f(z,\omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) a_n z^n, \tag{36}$$

where $\xi_n(\omega) = \exp\{2\pi i \alpha_n(\omega)\}$ be Steinhaus random variables, that is $\{\alpha_n(\omega)\}$ is a sequence of independent randon variables uniformly distributed on $[0; 1]$. Let $M_f(r,\omega) = \max\{|f(z,\omega)|\colon |z| = r\}$ be the maximum modulus of the function (36).

Then for every nonconstant entire function there exist a set $E \subset [1; +\infty)$ of finite logarithmic measure such that almost surely

$$0 \leq \varlimsup_{\substack{r\to+\infty \\ r\notin E}} \frac{\ln \mathfrak{M}_f(r) - \ln M_f(r,\omega)}{\ln\ln M_f(r,\omega)} \leq \frac{1}{4}. \tag{37}$$

Remark, that constants 0 and $1/4$ in previous inequality are sharp.

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QUANTITATIVE REGULARITY AT LEBESGUE POINTS OF THE BELTRAMI COEFFICIENT

It is well known that a quasiconformal mapping is differentiable almost everywhere and that almost every point is a Lebesgue point of its Beltrami coefficient. These two full-measure properties are not pointwise equivalent: a Lebesgue point of the coefficient need not be a differentiability point of the mapping, nor need a differentiability point of the mapping be a Lebesgue point of the coefficient. The qualitative study of local behavior at Lebesgue points of the Beltrami coefficient was initiated by Gutlyanskii and Ryazanov and developed further with their collaborators; see [1] for an extensive account.

We prove that power-rate convergence in the Lebesgue-point condition for μ yields a pointwise $C^{1+\alpha}$ real-linear expansion of f.

Theorem 1. *Let $f\colon \Omega \to \mathbb{C}$ be a quasiconformal mapping with Beltrami coefficient μ, and let $z_0 \in \Omega$. Suppose that, for some $\mu_0 \in \mathbb{C}$ with $|\mu_0| < 1$ and some $\beta > 0$,*

$$\int_{|z-z_0|<r} |\mu(z) - \mu_0|\, dA(z) = O(r^{\beta+2}) \qquad (r \downarrow 0).$$

Then, for every $0 < \alpha < \beta/(\beta+2)$, there exists $a \in \mathbb{C} \setminus \{0\}$ such that, as $h \to 0$,

$$f(z_0 + h) = f(z_0) + a\big(h + \mu_0 \overline{h}\big) + O(|h|^{1+\alpha}).$$

If $\mu_0 = 0$, this is the pointwise conformal expansion $f(z_0 + h) = f(z_0) + ah + O(|h|^{1+\alpha})$, with $a \neq 0$.

The hypothesis is a quantitative β-Lebesgue condition, since the averaged L^1-deviation of μ from μ_0 is $O(r^\beta)$. When $\mu_0 = 0$, the conclusion gives pointwise $C^{1+\alpha}$ conformality; this special case has a direct proof and also follows, via a standard dyadic estimate, from Shishikura's $C^{1+\alpha}$-conformality criterion [2]. We also discuss the interplay between differentiability of a quasiconformal mapping and quantitative Lebesgue behavior of its Beltrami coefficient, together with refinements based on Teichmüller's Modulsatz [3] in the setting of μ-homeomorphisms.

R E F E R E N C E S

1. Bojarski, B., Gutlyanskii, V., Martio, O., Ryazanov, V. *Infinitesimal Geometry of Quasiconformal and Bi-Lipschitz Mappings in the Plane*, EMS Tracts in Mathematics **19**, European Mathematical Society, Zürich (2013).
2. Shishikura, M. *Conformality of quasiconformal mappings at a point, revisited*, Ann. Acad. Sci. Fenn. Math. **43**, 981–990 (2018).
3. Teichmüller, O. *Untersuchungen über konforme und quasikonforme Abbildung*, Deutsche Math. **3**, 621–678 (1938).

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SCHWARZ LEMMA FOR HARMONIC MAPS

Let $n \geq 2$, let $F : \mathbb{D} \to H$ (in particular $F : \mathbb{D} \to \mathbb{B}^n$) be harmonic, and write $F = P[\Phi]$, where $\Phi \in h^\infty(\mathbb{D}, H)$ ($\Phi : \mathbb{T} \to \overline{\mathbb{B}^n}$) is the radial boundary function. Suppose that $F(0) = p$ and that dF_0 is conformal, meaning that $F_x(0)$ and $F_y(0)$ are orthogonal and have the same Euclidean norm. If $dF_0 = 0$, the desired estimate is trivial. Otherwise let

$$\Lambda = dF_0(\mathbb{R}^2)$$

be the image plane, and let $\lambda > 0$ be the conformal factor. The plane Λ is a linear two-plane through the origin in the ambient space $\mathbb{R}^n$. The point $p = F(0)$ need not lie in Λ; throughout this section, $\pi_\Lambda p$ denotes the orthogonal projection of the position vector p onto this direction plane. Write

$$p = p_\Lambda + p_\perp, \qquad p_\Lambda = \pi_\Lambda p \in \Lambda, \qquad p_\perp \perp \Lambda.$$

Choose an orthonormal basis e_1, e_2 of Λ with $p_\Lambda = ae_1$, where $a = |p_\Lambda|$; if $a = 0$, choose e_1 arbitrarily. Since dF_0 is a similarity onto Λ, a rotation of the source variable gives the normalization

$$F_x(0) = \lambda e_1, \qquad F_y(0) = \lambda e_2. \tag{38}$$

Put $b = |p_\perp|$, and if $F : \mathbb{D} \to \mathbb{B}^n$ set

$$c = \sqrt{1 - b^2}, \qquad s = \frac{a}{c}. \tag{39}$$

Consider the standard Möbius transformation $\varphi_s(z) = \frac{s+z}{1+sz}$ for $|z| = 1$, $0 \leq s < 1$, and define the boundary-support vector field

$$E(z) = E_s^c(z) = \big(c\varphi_s(z), p_\perp\big) \in \Lambda \oplus \Lambda^\perp, \quad c > 0, \tag{40}$$

where the first component is understood as a vector in Λ. Define the total weighted support integral:

$$I = I(\Phi) = \langle \Phi, E \rangle_w = \int_{\mathbb{T}} w(z) \langle \Phi(z), E(z) \rangle \, dm(z). \tag{41}$$

Note that $E_s^c(0) = p$ if and only if $cs = a$.

Let $\mathbb{D}$ denote the unit disk and $\rho_{\mathbb{D}}$ the hyperbolic metric on it.

Theorem 1. *Text of Theorem 1*
Let H be a real or complex Hilbert space, and let $F : \mathbb{D} \to H$ be a harmonic map belonging to the Hardy space $h^\infty(\mathbb{D}, H)$. Let $F(0) = p$, and suppose that the real differential dF_0 is conformal (or more generally, K-quasiconformal) and the tangent plane T_p is parallel to $\mathbb{D}$. If $dF_0 \neq 0$, let $\Lambda = dF_0(\mathbb{R}^2)$ be the closed two-dimensional image plane in H, and let $\lambda = \|dF_0\|$ be the conformal factor and assume that $|\langle \Phi, E_s^c \rangle_w| \leq 1 + s^2$ (the support inequality w.r.t. Möbius transformation E_s^c). If in addition $E_s^c(0) = p$ (i.e. $cs = a$), then E_s^c is the unique extremal, and we obtain:

$$\lambda \leq \lambda_0 = \frac{1 - |p|^2}{\sqrt{1 - |p|^2 + |\pi_\Lambda p|^2}}. \tag{42}$$

For $p \in \mathbb{B}^n$ and $v \in \mathbb{R}^n \simeq T_p\mathbb{B}^n$, define

$$CK_{\mathbb{B}^n}(p;v) = \frac{\sqrt{(1-|p|^2)|v|^2 + \langle p,v\rangle^2}}{1-|p|^2}. \tag{43}$$

This is the infinitesimal norm of the Beltrami–Klein model of real hyperbolic space. Let $\mathbb{B}^n_H$ denote the unit ball in Hilbert space H. We use this definition also for $p \in \mathbb{B}^n_H$ and $v \in H$, and by CK we also denote the Cayley–Klein metric on $\mathbb{B}^n_H$.

Theorem 2. *Let $F : \mathbb{D} \to \mathbb{B}^n_H$ be harmonic, let $z_0 \in \mathbb{D}$, and suppose that dF_{z_0} is K-quasiconformal at z_0. Then, for every $v \in T_{z_0}\mathbb{D}$,*

$$CK_{\mathbb{B}^n}(F(z_0); dF_{z_0}v) \leq K\frac{|v|}{1-|z_0|^2}. \tag{44}$$

Equivalently, in the Schwarz–Pick normalization,

$$CK_{\mathbb{B}^n}(F(z_0); dF_{z_0}v) \leq \rho_{\mathbb{D}}(z_0; v).$$

From **Theorem 2** we derive:

Theorem 3. *Let $F : \mathbb{D} \to \mathbb{B}^n_H$ be a harmonic K-quasiconformal map, then*

$$CK_{\mathbb{B}^n}(F(z_0), F(z)) \leq K\rho_{\mathbb{D}}(z_0, z).$$

The part of the plan of communication is joint work with M. Knežević.

R E F E R E N C E S

1. M. Mateljević, Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions, *J. Math. Anal. Appl.* **464** (2018), no. 1, 78–100.

2. M. Mateljević and M. Svetlik, Hyperbolic metric on the strip and the Schwarz lemma for HQR mappings, *Applicable Analysis and Discrete Mathematics* **14** (2020), no. 1, 150–168.

3. Mateljević, M., Mutavdžić, N. *The Boundary Schwarz Lemma for Harmonic and Pluriharmonic Mappings and Some Generalizations.* Bull. Malays. Math. Sci. Soc. 45, 3177–3195 (2022).

4. W. Zwonek, Complex geodesics in tube domains and their role in the study of harmonic mappings in the disc, *Anal. Math.* **48** (2022), no. 2, 601–617.

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CONTINUOUS EXTENSIONS OF OPEN DISCRETE MAPPINGS WITH INTEGRAL CONSTRAINTS

We consider open discrete mappings which are not assumed to preserve the boundary; an approach to the study of such mappings was developed in [1]. The boundary behavior of these mappings and the behavior of the corresponding families in the closure of the domain are studied.

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a domain, and let $n-1 < \alpha \leq n$. A Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is admissible for a family Γ of paths if

$$\int_\gamma \rho(x)\,|dx| \geq 1$$

for every locally rectifiable path $\gamma \in \Gamma$. In this case, we write $\rho \in \operatorname{adm}\Gamma$. The α-modulus of Γ is

$$M_\alpha(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int_{\mathbb{R}^n} \rho^\alpha(x)\,dm(x).$$

For $x_0 \in \overline{D} \setminus \{\infty\}$ and $0 < r_1 < r_2$, denote

$$S_i = S(x_0, r_i), \qquad i = 1, 2,$$

and

$$A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}.$$

Let $f : D \to \mathbb{R}^n \setminus F$ be an open discrete mapping. For every $x_0 \in D$, suppose that there exists $R_0 = R_0(x_0) > 0$ such that

$$M_\alpha\big(f(\Gamma(S_1, S_2, D))\big) \leq \int_{A(x_0, r_1, r_2) \cap D} Q_f(x)\eta^\alpha(|x - x_0|)\,dm(x) \qquad (1)$$

whenever $0 < r_1 < r_2 < R_0$ and $\eta : (r_1, r_2) \to [0, \infty]$ is a Lebesgue measurable function satisfying

$$\int_{r_1}^{r_2} \eta(r)\,dr \geq 1.$$

Here $Q_f : D \to [0, \infty]$ is a Lebesgue measurable function.

Let E be a closed subset of D, and let E_* be a closed subset of $\mathbb{R}^n$. We require that

$$C(f, \partial D) \subset E_*, \qquad f^{-1}(E_*) \subset E.$$

Assume that E is nowhere dense in D and that D is finitely connected on E.

For each $x_0 \in \partial D$, there is a finite integer $m = m(x_0) \geq 1$ such that every neighborhood U of x_0 contains a neighborhood $V \subset U$ for which $V \cap D$ is connected and $(V \cap D) \setminus E$ consists of at most m components.

For every component K of $f(D) \setminus E_*$, suppose that there exists a continuum $K_f \subset K$ such that

$$h(K_f) \geq \delta, \qquad h\big(f^{-1}(K_f), \partial D\big) \geq \delta,$$

where $\delta > 0$ is fixed.

Let $\Phi : [0, \infty] \to [0, \infty]$ be an increasing convex function. We consider mappings for which

$$\int_D \Phi(Q_f(x)) \frac{dm(x)}{(1+|x|^2)^n} \leq M, \tag{2}$$

where $M > 0$ is fixed. Integral constraints of this type were studied in [2].

Denote by $\mathcal{B}$ the family of all open discrete mappings $f : D \to \overline{\mathbb{R}^n} \setminus F$ for which there exists a Lebesgue measurable function $Q = Q_f : D \to [0, \infty]$ satisfying the condition (1) at every point $x_0 \in D$ and (2), together with the conditions on $C(f, \partial D)$, $f^{-1}(E_*)$, and the continua K_f stated above.

The following result is an analogue of the classical equicontinuity theorem of Näkki and Palka [3].

Theorem 1. *Let F be a set of positive capacity if $\alpha = n$, and let it be an arbitrary closed set if $n - 1 < \alpha < n$. Suppose that*

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{1/(\alpha-1)}} = \infty \tag{3}$$

for some $\delta_0 > \Phi(0)$.

If, over all $f \in \mathcal{B}$, the components of $f(D) \setminus E_*$ form an equi-uniform family with respect to α-modulus, then every mapping $f \in \mathcal{B}$ admits a continuous extension to ∂D. The corresponding family of extended mappings is equicontinuous in $\overline{D}$.

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R E F E R E N C E S

1. Desyatka, V., Sevost’yanov, E. *On boundary-non-preserving mappings with Poletsky inequality*, Canadian Math. Bull. **68**(3), 834–855 (2025).

2. Sevost’yanov, E. *On global behavior of mappings with integral constraints*, Anal. Math. Phys. **12**(3) (2022).

3. NakkiPalka Näkki, R., Palka, B. *Uniform equicontinuity of quasiconformal mappings*, Proc. Amer. Math. Soc. **37**(2), 427–433 (1973).

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FINITE ULTRAMETRIC SPACES WITH MAXIMUM CARDINALITY OF THE CENTER OF DISTANCES

A metric space (X, d) is called an *ultrametric space* if the *strong triangle inequality*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

holds for all x, y, $z \in X$.

Let (X, d) be a metric space and let $D(X)$ be the *distance set* of (X, d), $D(X) := \{d(x, y) : x, y \in X\}$. The *center of distances* of (X, d) is a subset $C(X)$ of the distance set $D(X)$, defined as the set of numbers $t \in D(X)$ such that for each $p \in X$ there exists $x \in X$ for which $d(p, x) = t$.

The concept of the center of distances was introduced by Wojciech Bielaś, Szymon Plewik, and Marta Walczyńska in [1]. This concept was used in [1] to generalize John von Neumann's theorem on permutations of two sequences with the same set of cluster points [2]. The centers of distances of ultrametric spaces generated by labeled trees were studied in [3]. Several interesting properties of the center of distances were established in [4–8].

The following theorem is the main result of [9].

Theorem 1. *Let $n \geq 1$ be an integer number, let $\log_2(n)$ be the binary logarithm of n, and let $\lfloor \log_2(n) \rfloor$ be the integer part of $\log_2(n)$. Then the inequality*

$$|C(X)| \leq 1 + \lfloor \log_2(n) \rfloor$$

holds for each ultrametric space (X, d) with $|X| = n$. Moreover, there exists an ultrametric space (Y, ρ) such that $|Y| = n$ and

$$|C(Y)| = 1 + \lfloor \log_2(n) \rfloor.$$

Theorem 1 implies the following corollary.

Corollary 2. *The inequality*

$$|X| \geq 2^{|C(X)|-1}$$

holds for each finite ultrametric space (X, d).

It is easy to see that isometric ultrametric spaces have the same center of distances. The following theorem claims, in particular, that the converse also holds under suitable restrictions on the spaces and their centers of distances [10].

Theorem 3. *Let (X, d) and (Y, ρ) be finite ultrametric spaces satisfying the equalities*

$$|X| = 2^{|C(X)|-1} \quad \textit{and} \quad |Y| = 2^{|C(Y)|-1}.$$

Then the following statements are equivalent.

(i) $C(X) = C(Y)$.

(ii) $D(X) = D(Y)$.

(iii) (X, d) *and* (Y, ρ) *are isometric.*

The following result from [10] is motivated by the work of Mateusz Kula on the center of distances of Bernstein subsets of $\mathbb{R}$ [11].

Proposition 4. *Let A be a finite subset of $\mathbb{R}^+$ and let $0 \in A$. Then, up to isometry, there exists a unique finite ultrametric space (X, d) such that*

$$|X| = 2^{|C(X)|-1} \quad \textit{and} \quad D(X) = C(X) = A.$$

R E F E R E N C E S

1. Bielas, W., Plewik, S., Walczyńska, M. *On the center of distances*, European Journal of Mathematics **4**, 687–698 (2018).

2. Neumann, J. *Charakterisierung des Spektrums Eines Integraloperators*, Hermann, Paris (1935).

3. Dovgoshey, O., Rovenska, O. *Center of distances of ultrametric spaces generated by labeled trees*, Mathematics **14**(5) (2026).

4. Bartoszewicz, A., Glab, S., Filipczak, M., Prus-Wisniowski, F., Swaczyna, J. *On generating regular cantorvals connected with geometric cantor sets*, Chaos, Solitons & Fractals **114**, 468–473 (2018).

5. Bartoszewicz, A., Glab, S., Marchwicki, J. *Recovering purely atomic finite measure from its range*, Journal of Mathematical Analysis and Applications **467**(2), 825–841 (2018).

6. Banakiewicz, M. *The center of distances of central Cantor sets*, Results in Mathematics **78**, 234 (2023).

7. Kula, M., Nowakowski, P. *Achievement sets of series in* $\mathbb{R}^2$, Results in Mathematics **79**, 221 (2024).

8. Nowakowski, P., Prus-Wiśniowski, F., Turoboś, F. *Spectre operator, achievement sets and sets of P-sums in a hyperspace of compact sets*, arXiv:2512.11803 (2025).

9. Dovgoshey, O., Rovenska, O. *On the center of distances of finite ultrametric spaces*, arXiv:2603.25850.

10. Dovgoshey, O., Jaiswal, N. S., Kainth, S. P. S., Kaur, G., Rovenska, O. *Maximal Center of Distances of Finite Ultrametric Spaces and Binary Trees*, (manuscript in preparation).

11. Kula, M. *Center of distances and Bernstein sets*, Real Analysis Exchange **50**, 1, 193–198 (2025).

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ON WEAKLY $(n-1)$-CONVEX SETS AND $(n-1)$-CONVEX HULLS IN LOW-DIMENSIONAL EUCLIDEAN SPACES

We study generalized convex sets in the real n-dimensional Euclidean space for $n = 2, 3$ known as weakly $(n-1)$-convex sets, and the closely related but not identical $(n-1)$-convex sets.

Recall that an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$, $n \geqslant 2$, is called a *hyperplane*. In $\mathbb{R}^2$, a hyperplane is a *straight line*. A hyperplane in $\mathbb{R}^3$ is a *plane*.

An open set in $\mathbb{R}^n$ is called *weakly $(n-1)$-convex* if through every boundary point of the set there passes a hyperplane that does not intersect the set. A closed set in $\mathbb{R}^n$ is *weakly $(n-1)$-convex* if it can be approximated from the outside by a countably decreasing family of open weakly $(n-1)$-convex sets. A set in $\mathbb{R}^n$ is *$(n-1)$-convex* if through every point in the complement of the set (with respect to the entire space) there passes a hyperplane that does not intersect the given set. For a subset $E \subset \mathbb{R}^n$, the intersection of all $(n-1)$-convex sets containing E is called its *$(n-1)$-convex hull* and is denoted by $\mathrm{conv}_{n-1}E$. A point in the complement of a set in $\mathbb{R}^n$ is called an *$(n-1)$-nonconvexity point* of the set if every hyperplane passing through the point intersects the set. The $(n-1)$-nonconvexity-point set corresponding to a subset $E \subset \mathbb{R}^n$ is denoted by $E^{\triangle}_{n-1}$.

Theorem 1. *Let $E \subset \mathbb{R}^2$ be an open weakly 1-convex subset and $E^{\triangle}_1 \neq \varnothing$. Then*

(a) *$E^{\triangle}_1$ is open and weakly 1-convex;*

(b) *the components of $E^{\triangle}_1$ are convex;*

(c) *any connected component of $\partial E^{\triangle}_1$ consisting of only smooth points is either a line segment, or a ray, or a single point;*

(d) *each component of $E^{\triangle}_1$ is the interior of a convex polygon if E is bounded and consists of a finite number of components.*

Theorem 2. *Let $E \subset \mathbb{R}^3$ be an open weakly 2-convex subset and $E^{\triangle}_2 \neq \varnothing$. Then*

(a) *$E^{\triangle}_2$ is open and weakly 2-convex;*

(b) *the components of $E^{\triangle}_2$ are convex;*

(c) *for any bounded component $\widetilde{E^{\triangle}_2}$ of $E^{\triangle}_2$ and any plane L such that $L \cap \widetilde{E^{\triangle}_2} \neq \varnothing$, the intersection $L \cap \partial\widetilde{E^{\triangle}_2}$ does not consist of only smooth points of $\widetilde{E^{\triangle}_2}$;*

(d) *there exists a bounded open weakly 2-convex set with a finite collection of components, and such that $E^{\triangle}_2$ is open, bounded, and distinct from the interior of a convex polyhedron.*

Theorem 3. *Let $E \subset \mathbb{R}^n$, $n = 2, 3$, be a closed subset such that $\mathrm{Int}\, E \neq \varnothing$. If E is weakly $(n-1)$-convex, then $\mathrm{Int}\, E$ is weakly $(n-1)$-convex.*

Theorem 4. *For any open subset $E \subset \mathbb{R}^n$, $n = 2, 3$, the $(n-1)$-convex hull* $\mathrm{conv}_{n-1}E$ *is open.*

Theorem 5. *For any subset $E \subset \mathbb{R}^n$, the following equality holds:*

$$\mathrm{conv}_{n-1}E = E \cup E^{\triangle}_{n-1}.$$

R E F E R E N C E S

1. Osipchuk, T.M. *Topological and Geometric Properties of the Set of 1-Nonconvexity Points of a Weakly 1-Convex Set in the Plane*, Ukrainian Mathematical Journal, **73**(12), 1918-1936 (2022).

2. Osipchuk, T.M., Tkachuk, M.V. *On weakly 1-convex sets in the plane*, Proceedings of the International Geometry Center, **16**(1), 42-49 (2023).

3. Osipchuk, T.M. *On weakly* 1*-convex and weakly* 1*-semiconvex sets in real Euclidean spaces*, Proceedings of the International Geometry Center, **18**(1), 102-117 (2025).

4. Osipchuk, T.M. *On weakly* $(n-1)$*-convex sets in low-dimensional Euclidean spaces*, Ukrainian Mathematical Journal (2026) (accepted for publication).

5. Osipchuk, T.M. *On weakly two-convex sets in three-dimensional Euclidean space*, Proceedings of the International Geometry Center, **19**(2), (2026) (accepted for publication).

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ESTIMATES OF THE SOLID MODULUS OF CONTINUITY OF THE LOGARITHMIC DOUBLE LAYER POTENTIAL

Let γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$, and let D^+ and D^- be the interior and exterior domains bounded by γ, respectively.

We consider the logarithmic double layer potential

$$\widehat{g}(z) := \frac{1}{2\pi}\int\limits_{\gamma} g(t)\frac{\partial}{\partial \mathbf{n}_t}\left(\ln\frac{1}{|t-z|}\right) ds_t = \mathrm{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{g(t)}{t-z}\,dt\right) \qquad \forall\, z\in D^{\pm},$$

where $\mathbf{n}_t$ and s_t denote the unit vector of the outward normal to the curve γ at a point $t\in\gamma$ and an arc coordinate of this point, respectively, and the integral density $g:\gamma\to\mathbb{R}$ takes values in the set of real numbers $\mathbb{R}$.

We consider an *Ahlfors-regular* curve (see [1, 2]), i.e., a closed rectifiable Jordan curve γ satisfying the condition $\sup\limits_{\xi\in\gamma}\mathrm{mes}\,\gamma_\varepsilon(\xi)=O(\varepsilon),\ \varepsilon\to 0$, where $\gamma_\varepsilon(\xi):=\{t\in\gamma: |t-\xi|\le\varepsilon\}$ and mes denotes the linear Lebesgue measure on the curve γ. The necessary and sufficient condition

$$M_\gamma(g,\varepsilon) := \sup_{\xi\in\gamma}\ \sup_{\delta\in(0,\varepsilon)}\left|\ \int\limits_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)}\big(g(t)-g(\xi)\big)\,d\arg(t-\xi)\right| \to 0\,, \quad \varepsilon\to 0\,, \qquad (1)$$

for the continuous extension of the function $\widehat{g}$ from the domain $D^{\pm}$ to the Ahlfors-regular boundary γ is established in [3]. Continuous extensions of the function $\widehat{g}$ to the closures $\overline{D^{\pm}}$ is denoted by $\widehat{g}^{\pm}$.

For a function $f\colon E\to\mathbb{C}$ continuous on $E\subset\mathbb{C}$, we use its modulus of continuity

$$\omega_E(f,\varepsilon) := \sup_{t_1,t_2\in E\,:\,|t_1-t_2|\le\varepsilon}|f(t_1)-f(t_2)|\,.$$

An upper estimate for the solid modulus of continuity of the function $\widehat{g}^{\pm}$ on the set $\overline{D^{\pm}}$ is established in the following theorem.

Theorem 1. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon\gamma\to\mathbb{R}$ be continuous on γ. If condition (1) is satisfied, then the following estimate is true:*

$$\omega_{\overline{D^{\pm}}}\big(\widehat{g}^{\pm},\varepsilon\big) \le c\left(M_\gamma(g,\varepsilon)+\varepsilon\int\limits_{\varepsilon}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}\,d\eta\right) \qquad \forall\,\varepsilon\in(0,d/8]\,, \qquad (2)$$

where $d:=\max\limits_{t_1,t_2\in\gamma}|t_1-t_2|$ and the constant c depends only on γ but does not depend on ε.

Consider a *Král curve* (see [3, 4]), i.e., an Ahlfors-regular curve γ satisfying the condition $\sup\limits_{\xi\in\gamma}V_\gamma[\arg(t-\xi)]<\infty$, where $V_\gamma[\arg(t-\xi)]$ is the variation of the function $\arg(t-\xi)$ on $\gamma\setminus\{\xi\}$.

In the case where the integration curve γ is a Král curve, condition (1) is satisfied, and estimate (2) is specified in the following theorem.

Theorem 2. *Let a closed Jordan curve γ be a Král curve and let a function $g\colon \gamma \to \mathbb{R}$ be continuous on γ. Then the following estimate is true:*

$$\omega_{\overline{D^{\pm}}}(\widehat{g}^{\pm}, \varepsilon) \le c\varepsilon \int\limits_{\varepsilon}^{2d} \frac{\omega_\gamma(g,\eta)}{\eta^2}\, d\eta \qquad \forall\, \varepsilon \in (0, d/8]\,, \tag{3}$$

where the constant c depends only on γ but does not depend on ε.

It is also proved that estimates (2) and (3) are exact with respect to the order of smallness as $\varepsilon \to 0$.

Special cases of Theorems 1 and 2, including estimates for the moduli of continuity on the integration curve, were obtained jointly with A.A. Sarana.

The talk is dedicated to the memory of Professor Oleg Gerus from Zhytomyr, Ukraine.

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R E F E R E N C E S

1. Ahlfors, L. *Zur theorie der überlagerungsflächen*, Acta Math. **65**, 157–194 (1935).

2. Salaev, V.V. *Direct and inverse estimates for a singular Cauchy integral along a closed curve*, Mathematical Notes of the Academy of Sciences of the USSR **19** (3), 221–231 (1976).

3. Plaksa, S.A. *Continuous extension to the boundary of a domain of the logarithmic double layer potential*, Eurasian Math. J. **16** (4), 54–75 (2025).

4. Král, J. *On the logarithmic potential of the double distribution*, Czech. Math. J. **14** (2), 306–321 (1964).

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ON THE COMPLETENESS OF SYSTEMS OF BESSEL FUNCTIONS OF POSITIVE HALF-INTEGER INDEX

Let $m \in \mathbb{N}$, let $L^2((0;1);x^{2m}dx)$ be the weighted Lebesgue space of all measurable functions $f:(0;1)\to\mathbb{C}$ satisfying $\int_0^1 x^{2m}|f(x)|^2\,dx<+\infty$, let ([1])

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)},\quad z\in\mathbb{C},$$

be the Bessel function of the first kind of index $\nu\in\mathbb{R}$, and let $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2\neq\rho_n^2$ for $k\neq n$. An entire function G is said to be of ([2]) exponential type $\sigma\in[0;+\infty)$ if for any $\varepsilon>0$ there exists a constant $c(\varepsilon)$ such that $|G(z)|\le c(\varepsilon)\exp((\sigma+\varepsilon)|z|)$ for all $z\in\mathbb{C}$. A system of elements $\{u_k : k\in\mathbb{N}\}$ in a separable Hilbert space $\mathcal{H}$ is called complete ([2]) if $\overline{\operatorname{span}}\{u_k : k\in\mathbb{N}\}=\mathcal{H}$.

We establish necessary and sufficient conditions for the completeness of system

$$\left\{\rho_k^m\sqrt{x\rho_k}J_{m+1/2}(x\rho_k) : k\in\mathbb{N}\right\}$$

in $L^2((0;1);x^{2m}dx)$ in terms of sequences of zeros of functions from certain classes $E_{2m,-}$ of odd entire functions of exponential type $\sigma\le 1$ (see Theorems 2–5). In this case, we obtain several analogues of the Paley-Wiener theorem for the Hankel type transform of order $m+1/2$ in $L^2((0;1);x^{2m}dx)$ related to this system ([3]).

Let $PW^2_{\sigma,-}$ be the class of all odd entire functions of exponential type $\sigma\in(0;+\infty)$ whose narrowing on $\mathbb{R}$ belongs to $L^2(\mathbb{R})$.

Theorem 1 ([3]). *Let $m\in\mathbb{N}$. An entire function $H_{m+1/2}$ has the representation*

$$H_{m+1/2}(z)=z^m\int_0^1\sqrt{tz}J_{m+1/2}(tz)t^{2m}h_{m+1/2}(t)\,dt \tag{45}$$

with some function $h_{m+1/2}\in L^2((0;1);x^{2m}dx)$ if and only if it is an odd entire function of exponential type $\sigma\le 1$ such that

$$H_{m+1/2}(0)=0,\quad \left(\frac{1}{z}\frac{d}{dz}\right)^s H_{m+1/2}(z)\bigg|_{z=0}=0,\quad s\in\{1;2;\dots;m-1\}, \tag{46}$$

and the function $\left(z^{-1}d/dz\right)^m H_{m+1/2}(z)$ belongs to $PW^2_{1,-}$. If these conditions are fulfilled, then

$$h_{m+1/2}(t)=\sqrt{\frac{2}{\pi}}\frac{1}{t^{3m}}\int_0^{+\infty}\sin(tz)\left(\frac{1}{z}\frac{d}{dz}\right)^m H_{m+1/2}(z)\,dz.$$

Let $m\in\mathbb{N}$, let $\widetilde{E}_{2m,-}$ be the class of entire functions $H_{m+1/2}$ that can be presented in the form (45) with some function $h_{m+1/2}\in L^2((0;1);x^{2m}dx)$, and let $E_{2m,-}$ be the class of odd entire functions $H_{m+1/2}$ of exponential type $\sigma\le 1$ satisfying (46) and the function $\left(z^{-1}d/dz\right)^m H_{m+1/2}(z)$ belongs to $PW^2_{1,-}$. Evidently, $\widetilde{E}_{2m,-}=E_{2m,-}$.

Theorem 2. *Let $m\in\mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of nonzero complex numbers such that $\rho_k^2\neq\rho_n^2$ for $k\neq n$. For a system $\left\{\rho_k^m\sqrt{x\rho_k}J_{m+1/2}(x\rho_k) : k\in\mathbb{N}\right\}$ to be incomplete in the space $L^2((0;1);x^{2m}dx)$ it is necessary and sufficient that a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$,* *where $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$, is a subsequence of zeros of some nonzero odd entire function $H_{m+1/2} \in E_{2m,-}$.*

Theorem 3. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ for $k \neq n$. Let a sequence $(\rho_k)_{k\in\mathbb{Z}\setminus\{0\}}$, where $\rho_{-k} := -\rho_k$, be a sequence of zeros of some odd entire function $G \notin E_{2m,-}$ of exponential type $\sigma \leq 1$ such that for $\alpha < 2-m$ holds*

$$|G(z)| \geq c_1(1+|z|)^{-\alpha}e^{|\operatorname{Im} z|}, \quad c_1 > 0, \quad \arg z = \pi/4 + j\pi/2, \quad j \in \{0;1;2;3\}.$$

Then the system $\left\{\rho_k^m\sqrt{x\rho_k}J_{m+1/2}(x\rho_k) : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^{2m}dx)$.

Theorem 4. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers. Assume that $|\rho_k| \leq \Delta k + \beta + \alpha_k$ for $0 < \Delta < \frac{\pi}{2+m\pi}$, $-\Delta < \beta < \Delta(1+2m)$, and a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that $\alpha_k \geq 0$, $\alpha_k = O(1)$ as $k \to +\infty$ and*

$$\sum_{k=1}^{\infty}|\alpha_{k+1}-\alpha_k| < +\infty, \quad \sum_{k=1}^{\infty}\frac{\alpha_k}{k} < +\infty.$$

Then the system $\left\{\rho_k^m\sqrt{x\rho_k}J_{m+1/2}(x\rho_k) : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^{2m}dx)$.

Theorem 5. *Let $m \in \mathbb{N}$ and $(\rho_k)_{k\in\mathbb{N}}$ be a sequence of distinct nonzero complex numbers such that $|\operatorname{Im}\rho_k| \geq \delta|\rho_k|$ for all $k \in \mathbb{N}$ and some $\delta > 0$. If a system $\left\{\rho_k^m\sqrt{x\rho_k}J_{m+1/2}(x\rho_k) : k \in \mathbb{N}\right\}$ is complete in $L^2((0;1);x^{2m}dx)$, then*

$$\sum_{k=1}^{\infty}\frac{1}{|\rho_k|} = +\infty.$$

We remark that, analogues of Theorems 1–5 for the Hankel type transform of order $-m-1/2$ with $m \in \mathbb{N}$ in $L^2((0;1);x^{2m}dx)$ were obtained in [4, 5].

R E F E R E N C E S

1. Watson, G.N. *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge (1944).

2. Levin, B.Ya. *Lectures on Entire Functions*, Transl. Math. Monogr. Amer. Math. Soc., Providence, R.I., **150** (1996).

3. Khats', R.V. *Paley-Wiener-type theorems for the Hankel transform of positive half-integer order in weighted L^2-spaces*, Res. Math. **33**(3), 43–55 (2025). doi:https://doi.org/10.15421/242526

4. Khats', R.V. *An analogue of the Paley-Wiener theorem for the Hankel type transform in weighted L^2-spaces*, Gulf J. Math. **17**(1), 1–11 (2024). doi:https://doi.org/10.56947/gjom.v17i1.1663

5. Khats', R.V. *Completeness of the systems of Bessel functions of negative half-integer index less than -1 in weighted L^2-spaces*, J. Class. Anal. **26**(1), 33–45 (2025). doi:https://doi.org/10.7153/jca-2025-26-03

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ON BOUNDARY VALUE PROBLEMS FOR GENERALIZED CAUCHY–RIEMANN EQUATIONS WITH SOURCES

In short, as we promissed in our last paper [1], we give in [2] a survey of the corresponding consequences from the well developed theory of the Beltrami equations in the complex plane $\mathbb{C}$ to generalized Cauchy-Riemann equations $\operatorname{grad} v = B \operatorname{grad} u + S$ in the real plane $\mathbb{R}^2$. By the physical interpretation, u represents a potential function of the incompressible fluid steady flow in anisotropic inhomogeneous media, that is reflected by matrix valued coefficients B, and v is a stream function. Thus, we are dealing here with the fundamental equations of hydromechanics (fluid mechanics) in anisotropic inhomogeneous media. The survey includes various types of results as theorems on existence, representation and regularity of solutions to the generalized Cauchy-Riemann equations with sources for the main boundary value problems of the potential theory of the type of Hilbert, Dirichlet, Neumann, Poincare and Riemann.

As it is well-known, the characteristic property of an analytic function $f = u + i\,v$ in the complex plane $\mathbb{C}$ is that its real and imaginary parts satisfy the **Cauchy-Riemann system**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,, \tag{47}$$

which can be also written as a single equation in the matrix form

$$\nabla v = \mathbb{H}\, \nabla u\,, \tag{48}$$

where ∇v and ∇u denotes the gradient of v and u, correspondingly, interpreted as vector-columns in $\mathbb{R}^2$, and $\mathbb{H} : \mathbb{R}^2 \to \mathbb{R}^2$ is the so-called **Hodge star operator** represented as the 2×2 matrix

$$\mathbb{H} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{49}$$

which carries out the counterclockwise rotation of vectors by the angle $\pi/2$ in the real plane $\mathbb{R}^2$. Note that $\mathbb{H}$ is an analog of the imaginary unit in the space $\mathbb{M}^{2\times2}$ of all 2×2 matrices with real entries because, for the unit matrix I,

$$\mathbb{H}^2 := -\mathrm{I}\,. \tag{50}$$

The article [2] is devoted to the study of the mentioned boundary value problems for **generalized Cauchy-Riemann equations with sources**

$$\nabla v = B\, \nabla u + S \tag{51}$$

with measurable coefficients $B : D \to \mathbb{M}^{2\times2}$ and sources $S : D \to \mathbb{R}^2$.

Hereafter $\mathbb{B}^{2\times2}$ denotes the subspace of $\mathbb{M}^{2\times2}$ of all matrices

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{52}$$

with $\det B = 1$, $b_{22} = -b_{11}$, $b_{12} < 0$ and $b_{21} > 0$. In this case $|\mu| < 1$, where

$$\mu = \mu_B := \frac{b_{12} + b_{21} - 2\,i\,b_{11}}{b_{12} - b_{21} - 2}\,. \tag{53}$$

In [2], we establish integral criteria in terms of the complex dilatation μ_B of the equations (51) with measurable matrix valued coefficients $B : D \to \mathbb{B}^{2\times 2}$ to have non-classical but regular solutions to the boundary value problems of Hilbert, Dirichlet, Neumann, Poincare and Riemann with suitable geometric interpretations of the corresponding boundary values. Moreover, we give the representation of these solutions (u, v) as a composition of the corresponding **generalized analytic pairs** (U, V) **with sources** Σ, i.e., continuous solutions in the Sobolev class $W^{1,1}_{\rm loc}$ for equations of the form

$$\nabla V \ = \ \mathbb{H} \, \nabla U \ + \ \Sigma \, , \tag{54}$$

and homeomorphic solutions of the associated Beltrami equations with the complex coefficient μ from (53) which have the hydrodynamic normalization at ∞.

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R E F E R E N C E S

1. Gutlyanski, V., Ryazanov, V., Salimov, A., Salimov, R. *On generalized Cauchy-Riemann equations for anisotropic and inhomogeneous media*, Journal of Mathematical Sciences, **288** (2), 199–224, (2025).

2. Gutlyanski, V., Ryazanov, V., Salimov, A., Salimov, R. *Boundary value problems for generalized Cauchy-Riemann equations with sources*, Ukrainian Mathematical Bulletin, **23** (1), 40-73, (2026); translated in Journal of Mathematical Sciences, **302** (3), (2026).

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ON CARATHÉODORY THEOREM ON RIEMANNIAN SURFACES

In what follows, the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$ have hyperbolic type with metrics $\widetilde{h}$ and $\widetilde{h}_*$, respectively (see, e.g., [1]). Besides that, $ds_{\widetilde{h}}$ and $d\widetilde{v}$, $ds_{\widetilde{h}_*}$ and $d\widetilde{v}_*$ denote the elements of length and area on the Riemannian surfaces $\mathbb{S}$ and $\mathbb{S}_*$, respectively. We set

$$\widetilde{B}(p_0, r) := \{p \in \mathbb{S} : \widetilde{h}(p, p_0) < r\}, \quad \widetilde{S}(p_0, r) := \{p \in \mathbb{S} : \widetilde{h}(p, p_0) = r\}. \tag{55}$$

A (maximal) dilatation of f at z is defined in local coordinates by the relation

$$K_f(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \tag{56}$$

for $J_f(z) \neq 0$, $K_f(z) = 1$ for $\|f'(z)\| = 0$ and $K_f(z) = \infty$ otherwise. It is not difficult to see that K_f does not depend on local coordinates because the transition mappings between two charts are conformal by the definition of the Riemannian surface.

Let D and D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, correspondingly. A mapping $f : D \to D_*$ is called a *mapping with finite distortion,* if $f \in W^{1,1}_{\rm loc}(D)$ and, in addition, there is a Lebesgue measurable $K(z)$, $K(z) < \infty$ a.e., such that $\|f'(z)\|^2 \leqslant K(z) \cdot J_f(z)$ for almost all $z \in D$. (Here $\|f'(z)\|$ and $J_f(z)$ denote the matrix norm of f' and the jacobian of f, respectively; see, e.g., [1]). A mapping $f : D \to D_*$ is called *discrete* if the preimage $f^{-1}(y)$ of any point $y \in D_*$ consists of isolated points only. A mapping $f : D \to D_*$ is called *open* if the image of any open set $U \subset D$ is an open set in D_*.

A boundary ∂G of the domain G is called *strongly accessible at the point* $p_0 \in \partial G$ if for each neighborhood U of p_0 there is a compactum $E \subset G$, a neighborhood $V \subset U$ of the same point and a number $\delta > 0$ such that $M(\Gamma(E, F, G)) \geqslant \delta$ for any continua E and F intersecting both ∂U and ∂V. We say that a boundary ∂G is *strongly accessible* if it is strongly accessible at each of its points. The following result holds.

Theorem. *Let D and D_* be domains in $\mathbb{S}$ and $\mathbb{S}_*$, correspondingly, which have compact closures $\overline{D} \subset \mathbb{S}$ and $\overline{D_*} \subset \mathbb{S}_*$, while ∂D has a finite number of components. Assume that $Q : \mathbb{S} \to (0, \infty)$ is a given function locally integrable with respect to the measure $\widetilde{v}$ on $\mathbb{S}$, $Q(p) \equiv 0$ in $\mathbb{S} \setminus D$. Let $f : D \to D_*$ be an open discrete mapping with a finite distortion in D such that $f(D) = D_*$ and $K_f(p) \leqslant Q(p)$ for almost all $p \in D$. In addition, assume that $C(f, \partial D) \subset E_*$ for some closed set $E_* \subset \mathbb{S}_*$ and $f^{-1}(E_*) = E$ for some closed subset $E \subset D$. Assume that:*

1) *the set E is nowhere dense in D, and D is finitely connected on E,*

2) *there is $1 \leqslant m < \infty$ such that, for any $P \in \overline{D}_P \setminus D$ and for every neighborhood U of P, there is a neighborhood $V \subset U$ of P such that $V \cap D$ is connected and $(V \cap D) \setminus E$ consists at most of m components,*

3) *all components of the set $D_* \setminus E_*$ have a strongly accessible boundary,*

4) *for any $p_0 \in \partial D$ there is $N \in \mathbb{N}$ and $r_0 = r_0(p_0) > 0$ such that the set $\widetilde{S}(p_0, r) \cap G$ consists of at most N components for any $0 < r < r_0$ whenever G is a component of $D \setminus E$.*

Then f has a continuous extension $f : \overline{D}_P \to \overline{D_*}$ by some metric ρ defined in $\overline{D}_P$ as a mapping between metric spaces $(\overline{D}_P, \rho)$ and $(\mathbb{S}_*, \widetilde{h}_*)$, while $\overline{f}(\overline{D}_P) = \overline{D_*}$, whenever at least one of the following conditions hold:

1) *the relations*
$$\int\limits_{\varepsilon}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} < \infty\,, \qquad \int\limits_{0}^{\varepsilon_0} \frac{dt}{\|Q\|(t)} = \infty\,, \tag{57}$$
hold at any $p_0 \in \partial D$, where $\|Q\|(t) := \int\limits_{\widetilde{S}(p_0,t)} Q(p)\, ds_{\widetilde{h}}(p)$,

2) *the condition $Q \in FMO(\partial D)$ holds.*

This result was accepted for print in [2].

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R E F E R E N C E S

1. Ryazanov, V., Volkov, S. *On the Boundary Behavior of Mappings in the Class $W^{1,1}_{loc}$ on Riemann Surfaces*, Complex Analysis and Operator Theory **11**, 1503–1520 (2017).

2. Sevost'yanov, E. *On Carathéodory theorem for Sobolev classes on Riemannian surface*, Ukrainian Math. Bull./Journal of Mathematical Science, 20 pages (accepted for print).

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SOME OPEN PROBLEMS IN FUNCTION THEORY OVER 35 YEARS OF MATEMATYCHNI STUDII (inspired by A.A. Goldberg, M.M. Sheremeta, Ya.V. Vasylkiv)

For over 35 years, the journal *Matematychni Studii* has been a leading forum for new discoveries in complex analysis and function theory. Since its foundation, prominent mathematicians have proposed numerous open problems that stimulated further research. In this abstract, we survey selected open problems formulated in the papers of A.A. Goldberg, M.M. Sheremeta, Ya.V. Vasylkiv, and the authors of present text.

Here we mention Goldberg's question from the first volume of *Matematychni Studii* [1]. Let E be the class of real entire functions of exponential type, all zeros of which are real, and the sequence of these zeros is bounded neither from the left nor from the right. Let $L^p(-\infty;+\infty)=\left\{f:\ \int_{-\infty}^{+\infty}|f(x)|^p dx<\infty\right\}$. Let the zeros of a function $f\in E$ be denoted as follows: $\dots<x_k<x_{k+1}<\dots$ (without taking into account their multiplicities), let $\xi_k\in(x_k,x_{k+1})$ be a zero of the function f', and let $b_k=|f(\xi_k)|$. A.A. Goldberg showed [1] that for $f\in L^p\cap E$, where $p\geq 1$, it is necessary, but it is not sufficient, that $\sum_{k=-\infty}^{\infty}b_k^p<\infty$. However, in a remark, he wrote that it was unable to clarify whether the condition $p\geq 1$ could be replaced by $p>0$.

Next, we address the theory of entire functions of bounded index. M.M. Sheremeta initiated the study of boundednesss of l-index for entire functions and functions analytic in the unit disc. A persistent open problem formulated by M.M. Sheremeta [2] is the complete characterization of the continuous positive functions l for which an entire function with given zeros has a bounded l-index. Although M.T. Bordulyak [3] pointed out that a necessary and sufficient condition for the existence of such a function is that the multiplicities zeros of the entire function are uniformly bounded, her proof – despite its constructive nature – says nothing about the properties of the positive continuous function she constructed. In particular, regarding its membership in the class $Q=\{l:\ l(r+O(1/l(r)))=O(l(r)), r=|z|\to+\infty\}$, which is necessary for many theorems in the theory.

In recent years, A.I. Bandura and O.B. Skaskiv extended this concept to functions of several complex variables. They investigated functions analytic in the unit polydisc $\mathbb{D}^n$ and the unit ball $\mathbb{B}^n$. Some unsolved problems in the theory of entire functions of several variables in connection with investigation of functions of bounded L-index in direction see in [4]. In particular, an open problem proposed by A.I. Bandura is to find necessary and sufficient conditions for an analytic function F in a unit ball to have a bounded $\mathbf{L}$-index in joint variables, independent of the specific directional vectors.

Theorem. *For every analytic function with bounded multiplicities of zeros at every slice in a given direction from the unit polydisc, there exists a positive continuous function* $\mathbf{L}$ *such that the primary analytic function has bounded* $\mathbf{L}$*-index in the same direction.*

Separately, we must highlight the fundamental problems investigated by Ya.V. Vasyl'kiv [5] in *Matematychni Studii* concerning the asymptotic behavior of Lebesgue integral means. In his works, particularly those co-authored with O.Z. Korenivs'ka, he studied the precise relations between the Lebesgue integral means of logarithms of analytic functions in the unit disc and their Nevanlinna characteristics. A challenging problem arising from these studies is finding the exact asymptotic estimates and sharp inequalities for $m_p(r, \log f) := \left(\frac{1}{2\pi}\int_0^{2\pi} |\log f(re^{i\theta})|^p d\theta\right)^{1/p}$ in terms of the Nevanlinna characteristic $T(R, f)$ for $1 \leq p < +\infty$. While sharp relations between the orders of these characteristics were successfully established for functions of finite order, fully describing the extremal functions that turn these asymptotic inequalities into precise equalities for arbitrary analytic functions in the unit disc remains a deep open question.

The problems discussed above represent only a small part of the open mathematical problems published in *Matematychni Studii* on function theory. Their solution will undoubtedly lead to new methodologies and insights in modern complex analysis.

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REFERENCES

1. A.A. Goldberg, *On a criterion of belonging of entire function of exponential type to to the class L^p*, Mat. Stud. **1**, 55-60 (1991). doi:10.30970/ms.1.55-60

2. M.M. Sheremeta, *Five open problems in the theory of entire functions*, Mat. Stud. **6**, 157-159 (1996).

3. M.T. Bordulyak, *A proof of a Sheremeta conjecture concerning entire function of bounded l-index*, Mat. Stud., **12** (1), 108-110 (1999).

4. A.I. Bandura, O.B. Skaskiv, *Open problems for entire functions of bounded index in direction*, Mat. Stud., **43** (1), 103-109 (2015).
doi:10.15330/ms.43.1.103-109

5. Ya.V. Vasyl'kiv, O.Z. Korenivs'ka, *On the relation between the Lebesgue integral means and Nevanlinna characteristic of analytic functions in the unit disc*, Mat. Stud. **36**(1), 58-64 (2011). doi:10.30970/ms.36.1.58-64

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POWER-TYPE DISTORTION ESTIMATES FOR SOLUTIONS OF THE BELTRAMI EQUATION WITH TWO CHARACTERISTICS

Let D be a domain in the complex plane $\mathbb{C}$ and let μ and $\nu\colon D \to \mathbb{C}$ be measurable functions with $|\mu(z)|+|\nu(z)| < 1$ a.e. (almost everywhere) in D. We consider the *Beltrami equation with two characteristics*

$$f_{\overline{z}} = \mu(z)f_z + \nu(z)\overline{f_z} \quad \text{a.e.} \quad \text{in} \quad D, \tag{58}$$

where $f_{\overline{z}} = (f_x + if_y)/2$, $f_z = (f_x - if_y)/2$, $z = x+iy$, f_x and f_y are the partial derivatives of f by x and y, respectively. The functions μ and ν are called the *complex coefficients* and

$$K_{\mu,\nu}(z) := \frac{1+|\mu(z)|+|\nu(z)|}{1-|\mu(z)|-|\nu(z)|}$$

is the *dilatation quotient* for the equation (58).

We use in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the *chordal distance* $h(z,\zeta) = |\pi(z) - \pi(\zeta)|$, where π is the stereographic projection of $\overline{\mathbb{C}}$ onto the sphere $S^2\left(\frac{1}{2}e_3, \frac{1}{2}\right)$ in $\mathbb{R}^3$:

$$h(z,\zeta) = \frac{|z-\zeta|}{(1+|z|^2)^{\frac{1}{2}}\,(1+|\zeta|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \neq \zeta,$$

$$h(z,\infty) = \frac{1}{(1+|z|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty.$$

The *chordal diameter* of a set $E \subset \overline{\mathbb{C}}$ is $h(E) = \sup\limits_{z_1,z_2 \in E} h(z_1,z_2)$.

Suppose that $z_0 \in D$ and a function $\varphi\colon D \to \mathbb{C}$ is integrable in a disk $B(z_0,\varepsilon_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon_0\}$, where $0 < \varepsilon_0 < \operatorname{dist}(z_0, \partial D)$. For a centering function $l\colon (0,\varepsilon_0] \to \mathbb{C}$ and for every $\varepsilon \in (0,\varepsilon_0]$, define the quantity

$$\omega_{\varphi,l}(z_0,\varepsilon) := \int\limits_{B(z_0,\varepsilon)} |\varphi(z) - l(\varepsilon)|\,dxdy.$$

We say that φ has a *finite mean l-oscillation at a point* z_0 and denote $\varphi \in FMO_l(z_0)$ if

$$\limsup_{\varepsilon\to 0} \frac{\omega_{\varphi,l}(z_0,\varepsilon)}{\pi\varepsilon^2} < \infty.$$

Theorem 1. *Let D be a domain in the complex plane $\mathbb{C}$, $z_0 \in D$, and let D' be a domain in $\overline{\mathbb{C}}$ with $h\left(\overline{\mathbb{C}} \setminus D'\right) \geqslant \Delta > 0$. Suppose that μ and $\nu\colon D \to \mathbb{C}$ are measurable functions with $|\mu(z)| + |\nu(z)| < 1$ a.e., $f\colon D \to D'$ is a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation (58) with $K_{\mu,\nu} \in FMO_l(z_0)$, and*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} |l(\varepsilon_k)| < \infty, \quad \varepsilon_k = e^{-k+1}\varepsilon_0,$$

where $0 < \varepsilon_0 < \sqrt{\delta_0}$, $\delta_0 = \min\{1/e, \operatorname{dist}(z_0, \partial D)\}$. *Then*

$$h\left(f(\zeta), f(z_0)\right) \leqslant \frac{32}{\Delta} \cdot |\zeta - z_0|^{\frac{1}{\sigma}}$$

for all $\zeta \in B\left(z_0, \varepsilon_0^2\right)$, *where* $\sigma = 2(d_0 e^2 + 2L)$ *and*

$$d_0 = \sup_{\varepsilon \in (0, \varepsilon_0]} \frac{1}{\pi \varepsilon^2} \int\limits_{B(z_0, \varepsilon)} |K_{\mu,\nu}(z) - l(\varepsilon)|\, dxdy, \quad L = \sup_{N \geqslant 1} \frac{1}{N} \sum_{k=1}^{N} |l(\varepsilon_k)|.$$

Recall that a point $z_0 \in D$ is called a *Lebesgue point* of a function $\varphi\colon D \to \mathbb{C}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int\limits_{B(z_0, \varepsilon)} |\varphi(z) - \varphi(z_0)|\, dxdy = 0.$$

Corollary 1. *Let* D *be a domain in the complex plane* $\mathbb{C}$, $z_0 \in D$, *and let* D' *be a domain in* $\overline{\mathbb{C}}$ *with* $h\left(\overline{\mathbb{C}} \setminus D'\right) \geqslant \Delta > 0$. *Suppose that* μ *and* $\nu\colon D \to \mathbb{C}$ *are measurable functions with* $|\mu(z)| + |\nu(z)| < 1$ *a.e.,* $f\colon D \to D'$ *is a homeomorphic* $W^{1,1}_{\mathrm{loc}}$ *solution of the Beltrami equation* (58), *and* z_0 *is a Lebesgue point of the function* $K_{\mu,\nu}$. *Then*

$$h\left(f(\zeta), f(z_0)\right) \leqslant \frac{32}{\Delta} \cdot |\zeta - z_0|^{\frac{1}{\kappa}}$$

for all $\zeta \in B\left(z_0, \varepsilon_0^2\right)$, *where* $0 < \varepsilon_0 < \sqrt{\delta_0}$, $\delta_0 = \min\{1/e, \operatorname{dist}(z_0, \partial D)\}$, $\kappa = 2(d_0 e^2 + 2|K_{\mu,\nu}(z_0)|)$, *and*

$$d_0 = \sup_{\varepsilon \in (0, \varepsilon_0]} \frac{1}{\pi \varepsilon^2} \int\limits_{B(z_0, \varepsilon)} |K_{\mu,\nu}(z) - K_{\mu,\nu}(z_0)|\, dxdy.$$

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KOEBE THEOREM FOR ORLICZ-SOBOLEV CLASSES

Let

$$l\,(f'(x)) \;=\; \min_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}\,,$$

$$J(x,f)=\det f'(x)\,,$$

and define for any $x\in D$

$$K_I(x,f) \quad = \quad \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n}, & J(x,f)\neq 0, \\ 1, & f'(x)=0, \\ \infty, & \text{otherwise} \end{cases}\,,$$

Let $Q:D\to[0,\infty]$ be a Lebesgue measurable function. Denote by $q_{x_0}(r)$ the integral average of $Q(x)$ under the sphere $|x-x_0|=r$,

$$q_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r} Q(x)\,d\mathcal{H}^{n-1}\,,$$

where ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$.

The definitions of the Orlicz-Sobolev classes $W^{1,\varphi}_{\rm loc}$ and finite mean oscillation (FMO) may be found in [1]. Given a continuum $E\subset D\subset\mathbb{R}^n$, $n\geqslant 3$, $\delta>0$, a non-decreasing function $\varphi:[0,\infty)\to[0,\infty)$ and a Lebesgue measurable function $Q:\mathbb{R}^n\to[0,\infty]$ we denote by $\mathfrak{F}^{\varphi,Q}_{E,\delta}(D)$ the family of all homeomorphisms $f:D\to\mathbb{R}^n$, such that $g:=f^{-1}$ belong to $W^{1,\varphi}_{\rm loc}(f(D))$ with $K_I(y,f^{-1})\leqslant Q(y)$ for almost all $y\in f(D)$ such that $h(f(E))\geqslant\delta$. The following statement is true.

Theorem 1. *Let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $B(x_0,\varepsilon_1):=\{x\in\mathbb{R}^n: |x-x_0|<\varepsilon_1\}\subset D$ for some $\varepsilon_1>0$. Assume that, φ satisfies the Calderon condition*

$$\int\limits_1^{\infty}\left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}}dt<\infty\,,$$

$Q\in L^1(\mathbb{R}^n)$ and, in addition, one of the following conditions hold:

1) $Q\in FMO(\overline{\mathbb{R}^n})$;

2) for any $y_0\in\overline{\mathbb{R}^n}$ there is $\delta(y_0)>0$ such that the relation

$$\int\limits_0^{\delta(y_0)}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}=\infty$$

holds. Then there is $r_0>0$, which does not depend on f, such that

$$f(B(x_0,\varepsilon_1))\supset B_h(f(x_0),r_0)\qquad \forall\ f\in\mathfrak{F}^{\varphi,Q}_{E,\delta}(D)\,,$$

where $B_h(f(x_0), r_0) = \{w \in \overline{\mathbb{R}^n} : h(w, f(x_0)) < r_0\}$.

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R E F E R E N C E S

1. Sevost'yanov, E. *Mappings with Direct and Inverse Poletsky Inequalities. Developments in Mathematics (DEVM, volume 78)*, Springer Nature Switzerland AG, Cham, New York (2023).

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SOBOLEV INEQUALITIES FOR DIFFERENTIAL FORMS

Sobolev inequality for differential forms extend classical to the setting of differential forms on manifolds. These inequalities relate the norms of differential forms and their exterior derivatives. These inequalities have applications to partial differential equations on manifolds in areas like Hodge theory and geometric analysis problems.

R E F E R E N C E S

1. Vladimir Gol'dshtein and Marc Troyanov, Sobolev Inequalities for Differential and Lq,p-Cohomology, The Journal of Geometric Analysis, Volume 16, Number 4, 2006, 597-630.

2. V.Gol'dshtein, Ya. Kopylov, R.Panenko. Some Poincare–Sobolev inequalities for differential forms. Journal of Mathematical Sciences, Volume 298, pages 532–549 (2026)

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UPPER MODULUS ESTIMATES IN ONE CLASS OF MAPPINGS

Let $x_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \quad B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

and

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,. \tag{59}$$

In what follows, M_p denotes the p-modulus of families of paths (or surfaces). The notion $\mathrm{ext}_p\,\mathrm{adm}$ may be found in [1]. Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 2$. A mapping $f : D \to D'$ is called a *lower Q-mapping at a point x_0 relative to p-modulus* if

$$M_p(f(\Sigma_\varepsilon)) \geqslant \inf_{\rho \in \mathrm{ext}_p\,\mathrm{adm}\Sigma_\varepsilon} \int\limits_{D \cap A(x_0, \varepsilon, r_0)} \frac{\rho^p(x)}{Q(x)}\, dm(x)$$

for $A(x_0, \varepsilon, r_0) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < r_0\}$, $r_0 \in (0, d_0)$, $d_0 = \sup\limits_{x \in D} |x - x_0|$, where Σ_ε is the family of all intersections of the spheres $S(x_0, r)$ with the domain D, $r \in (\varepsilon, r_0)$. Given a mapping $f : D \to \mathbb{R}^n$, we denote

$$C(f, x) := \{y \in \overline{\mathbb{R}^n} : \exists\, x_k \in D : x_k \to x, f(x_k) \to y, k \to \infty\}$$

and

$$C(f, \partial D) = \bigcup_{x \in \partial D} C(f, x)\,.$$

The definitions of a regular domain, prime ends space, cuts, nested domains etc. may be found in [1]. Denote by ω_{n-1} the area of the unit sphere $\mathbb{S}^{n-1} = S(0, 1)$. In what follows, we denote by E_D the set of all prime ends in D. The completion of the domain D by its prime ends is denoted by $\overline{D}_P$. Given $A, B \subset \mathbb{R}^n$, we set $d(A, B) = \inf\limits_{x \in A, y \in B} |x - y|$. The following statement holds, see [2].

Theorem 1. *Let $f : D \to \mathbb{R}^n$ be a bounded, open and discrete lower Q-mapping with respect to p-modulus, $Q \in L^{\frac{n-1}{p-n+1}}_{loc}(\mathbb{R}^n)$, $p > n - 1$ and $\alpha := \frac{p}{p-n+1}$. Let D be a regular domain, let $P_0 \in E_D$ and let d_m, $m = 1, 2, \ldots$, be a sequence of nested domains corresponding to P_0 such that the cuts σ_m corresponding to d_m, $m = 1, 2, \ldots$, lies on the spheres $S(x_0, r_m)$, where $x_0 \in \partial D$ and $r_m \to 0$ as $m \to \infty$. Let E be a closed set with respect to D, let E_* be a closed in $\overline{\mathbb{R}^n}$, $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, such that $C(f, \partial D) \subset E_*$ and $f^{-1}(E_*) = E$. Then, given $\varepsilon_1 > 0$ and $m \in \mathbb{N}$ there is $k_0 = k_0(m, \varepsilon_1) \in \mathbb{N}$ such that the relation*

$$M_\alpha(f(\Gamma(C_1, C_2, D \setminus E))) \leqslant \frac{\omega_{n-1}}{I_k^{\alpha-1}}\,,$$

$$I_k := \int\limits_{r_k}^{\varepsilon_1} \frac{dr}{r^{\frac{n-1}{\alpha-1}} \widetilde{q}_{x_0}^{\frac{1}{\alpha-1}}(r)}\,, \qquad \widetilde{q}_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{|x - x_0| = r} Q^{\frac{n-1}{p-n+1}}(x)\, d\mathcal{H}^{n-1}\,,$$

holds for any $k \geqslant k_0(m, \varepsilon_1)$, *every compactum* $C_2 \subset D \setminus (d_m \cup E)$ *such that* $d(f^{-1}(f(C_2)), x_0) \geqslant \varepsilon_1$ *and each continuum* $C_1 \subset d_k \setminus E$ *which contains in the same component of* $D \setminus E$ *as* C_2, *where* $A(x_0, r_k, \varepsilon_1) = \{x \in \mathbb{R}^n : r_k < |x - x_0| < \varepsilon_1\}$.

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R E F E R E N C E S

1. Sevost'yanov, E. *Mappings with Direct and Inverse Poletsky Inequalities. Developments in Mathematics (DEVM, volume 78)*, Springer Nature Switzerland AG, Cham, New York (2023).

2. Kovba, Zarina, Sevost'yanov, Evgeny. *On Caratheodory prime ends extension for unclosed Orlicz-Sobolev classes*, https://arxiv.org/abs/2604.15026.

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HARDY SPACES OF MAPPINGS

We discuss the Hardy spaces of quasiconformal and quasiregular mappings including their brief history and some recent developments by the speaker and coworkers. The presentation is based on the joint works with Katrin Fassler, Maria Gonzalez and Ivan Caamano.

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ENTIRE FUNCTIONS OF SEVERAL VARIABLES: BEHAVIOUR OF DIRECTIONAL DERIVATIVES

The present paper is devoted to the analogues of the main relation of the Wiman-Valiron theory for entire functions of several complex variables which are bounded in the half-spaces. We will prove counterparts of the second relation in (1) for the directional derivative of an entire multiple Dirichlet series with arbitrary non-negative exponents. We will obtain the relation as a consequence of the theorem on the behavior of the directional derivative of an entire function F of several complex variables $z = (z_1, \dots, z_p)$ at the neighborhood of a point w. This point w is chosen such that the value $F(w)$ is close to the supremum of its modulus on the topological boundary of polylinear domains.

The Wiman-Valiron theorem [2] states that for every entire function f, for small $|\eta|$ in some sense and for any point z, $|z| = r$, such that $|f(z)| = M_f(r)$, we have

$$f(ze^{\eta}) \sim e^{\eta\nu_f(r)} f(z), \qquad \frac{f^{(k)}(z)}{f(z)} \sim \left(\frac{\nu_f(r)}{z}\right)^k, \tag{60}$$

as $|z| = r \to +\infty$, $r \notin E$, and E is some set of finite logarithmic measure (f.l.m.), that is $\int_{E\cap[1,+\infty)} d\ln r < +\infty$.

Let us denote $K_f(r) = r(\ln M_f(r))'_+$, where $(\ln M_f(r))'_+$ is the righ-hand derivative. From the theorem obtained in [3] for the entire Dirichlet series, it follows that for each entire function f, $K_f(r) \sim \nu_f(r) \quad (r \to +\infty,\ r \notin E)$, where $E = E(f)$ is some set of f.l.m.

Let $\mathbb{R}^p$ and $\mathbb{C}^p$ $(p \in \mathbb{N})$ be real and complex p-dimensional vector spaces, respectively; $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, +\infty)$, $p \in \mathbb{N}$. For $a = (a_1, \dots, a_p) \in \mathbb{R}^p$, $b = (b_1, \dots, b_p) \in \mathbb{R}^p$ we write $a < b$ and $a \le b$, if $(\forall j, 1 \le j \le p)\colon a_j < b_j$, and $(\forall j, 1 \le j \le p)\colon a_j \le b_j$, respectively. For $z = (z_1, \dots, z_p) \in \mathbb{C}^p$, $w = (w_1, \dots, w_p) \in \mathbb{C}^p$, we denote $(z, w) = z_1w_1 + \ldots + z_pw_p$, $\|z\| = z_1 + \ldots + z_p$, $\|z\|_0 = z_1 + 2z_2 + \ldots + pz_p$, $\mathrm{Re}\ z = (\mathrm{Re}\ z_1, \dots, \mathrm{Re}\ z_p)$, and for $R = (r_1, \dots, r_p) \in \mathbb{R}^p$ we denote $\Pi_R = \{z \in \mathbb{C}^p\colon \mathrm{Re}\ z < R\}$.

Let $A = (A_1, \dots, A_p) \in \mathbb{R}^p \setminus \{0\}$ be a fixed vector. We say that the system of domains $\{G(r, A)\}_{r\ge0}$, $G(r, A) \subset \mathbb{C}^p$, is an exhaustion of the space $\mathbb{C}^p$ from the class Σ, if

a) $\bigcup\limits_{r\ge0} G(r, A) = \mathbb{C}^p$;

b) $G(r_1, A) \subset G(r_2, A)$ $(0 \le r_1 < r_2 < +\infty)$;

c) if $z = (z_1, \dots, z_p) \in G(r, A)$, then for every $y = (y_1, \dots, y_p) \in \mathbb{R}^p$ we get $z + iy = (z_1 + iy_1, \dots, z_p + iy_p) \in G(r, A)$;

d) if $z \in G(r, A)$, then $(z - rA) \in G(0, A)$;

e) for given $r \ge 0$ there exist $R_*, R^* \in \mathbb{R}^p$ such that $\Pi_{R_*} \subset G(r, A) \subset \Pi_{R^*}$.

We will denote the class of such exhaustions $\{G(r, A)\}_{r\ge0}$ by Σ.

Consider the class $\mathcal{H}^p$ of an entire functions in $\mathbb{C}^p$, that are bounded in an arbitrary domain Π_R, $R = (R_1, \dots, R_p) \in \mathbb{R}_+^p$. For a function $F \in \mathcal{H}^p$ and $x \in \mathbb{R}^p$ it is obvious that

$$M(x, F) := \sup\{|F(x + iy)| : y \in \mathbb{R}^p\} < +\infty, \tag{61}$$

and also that if the analytic function F is bounded in a polylinear domain G, then for every x such that $\{z \in \mathbb{C}^p\colon \ \mathrm{Re}\ z_1 = x_1, \dots, \mathrm{Re}\ z_p = x_p\} \subset G$ one has $M(x, F) < +\infty$.

For a function $F \in \mathcal{H}^p$ and $r > 0$ we also denote

$$S_F(r, A) := \sup\{|F(z)|\colon z \in G(r, A)\} = \sup\{|F(z)|\colon z \in \partial G(r, A)\}. \tag{62}$$

Since $\ln S_F(r, A)$ is a convex function, it has a right-hand derivative everywhere $L_F(r, A) := \big(\ln S_F(r, A)\big)'_+$, which is a non-decreasing function.

Let $\mathcal{L}$ be the class of positive continuous functions which are increasing to $+\infty$ on $[0; +\infty)$.

Let $\mathcal{L}_2$ be the class of continuous positive non-decreasing on $[0; +\infty)$ functions h such that $h\left(x + \dfrac{1}{h(x)}\right) = O(h(x)) \quad (x \to +\infty)$.

Let h be a positive continuous non-decreasing function and $E \subset [0; +\infty)$ be a locally Lebesgue measurable set of finite measure with $\mathrm{meas}\, E = \int_E dx < +\infty$.
Then we define the asymptotic h-density of E as $D_h(E) = \varlimsup\limits_{R\to+\infty} h(R) \cdot \mathrm{meas}\,(E \cap [R, +\infty))$.

Theorem 1 ([1]). *Let $F \in \mathcal{H}^p$ and $A \in \mathbb{R}^p$ be such that $L_F(r, A) \uparrow +\infty$ $(r \to +\infty)$. If $\Phi \in \mathcal{L}$, $h \in \mathcal{L}_2$ satisfy conditions $u(r) \geq \Phi(r)\,(r \geq r_0)$, $\quad h(r) = o(\Phi(r)) \quad (r \to +\infty)$., then there exists a set $E \subset \mathbb{R}_+$ of zero asymptotic h-density (i.e. $D_h(E) = 0$) such that for each $k \in \mathbb{N}$ one has*

$$F_A^{(k)}(w) = (1 + o(1))\, L_F^k(r, A)\, F(w) \text{ as } r \to +\infty \ (r \in \mathbb{R}_+ \setminus E) \tag{63}$$

for every point $w \in \partial G(r, A)$ provided inequality $|F(w)| \geq S_F(r, A)/(1 + \varepsilon(r))$ with a given function $\varepsilon(r)$ which is decreasing to $+0$ $(r \to +\infty)$.

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REFERENCES

1. Bandura, A.; Dubei, S.; Salo, T.; Skaskiv, O. *Entire Functions of Several Variables: Behaviour of Directional Derivatives.* Carpathian Math. Publ. 2026, The local growth of power series: a survey of the Wiman–Valiron method, 67-77.

2. Hayman W.K. *The local growth of power series: a survey of the Wiman–Valiron method.* Canad. Math. Bull. 1974, **17** (3), 317–358. doi:10.4153/CMB-1974-064-0

3. Skaskiv O., Bandura A., Salo T., Dubey S. *Entire functions of several variables: analogues of Wiman's theorem*, Axioms 2025, **14** (3), 216. doi: 10.3390/axioms14030216

Секція 5

Теорія ймовірностей та математична статистика / Probability Theory and Mathematical Statistics

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ABOUT CONVERGENCE OF STOCHASTIC EQUATIONS INVOLVING LOCAL TIME AND NON-REGULAR COEFFICIENTS

It is well known that the convergence of coefficients in an Itô's stochastic equation is insufficient for the weak convergence of its solutions; an additional condition is required.

This paper investigates the weak convergence of solutions to stochastic equations involving local time, specifically addressing cases with non-regular dependence of coefficients on a small parameter ε:

$$\xi_\varepsilon(t) = x + \beta_\varepsilon L^{\xi_\varepsilon}(t,0) + \int_0^t \big(b_\varepsilon(\xi_\varepsilon(s)) + g_\varepsilon(\xi_\varepsilon(s))\big)ds + \int_0^t \sigma_\varepsilon(\xi_\varepsilon(s))dw(s). \tag{64}$$

A stochastic equation involving local time of a process was first investigated in [1] and [2]. Moreover in [2], [3], [4] were obtained formulae which connect solutions of stochastic equations with local time of a process with solutions of Itô's stochastic equations.

We assume that the coefficients of stochastic equation (1) satisfy the following conditions: $\beta_\varepsilon \to \beta$ when $\varepsilon \to 0$, $|\beta_\varepsilon| < 1$ and $|\beta| < 1$, there exists a constant $\Lambda > 0$ such that $|g_\varepsilon(x)| < \Lambda$, $\dfrac{1}{\Lambda} < \sigma_\varepsilon^2(x) < \Lambda$ and for every $x \in \mathbb{R}$

$$\left|\int_0^x \frac{b_\varepsilon(y)}{\sigma_\varepsilon^2(y)}dy\right| \le \Lambda\,.$$

Denote by $\xi(t)$ a weak solution following stochastic equation involving a local time of a process (with $|\gamma| < 1$)

$$\xi(t) = x + \gamma L^{\xi}(t,0) + \int\limits_0^t g(\xi(s))ds + \int\limits_0^t \sigma(\xi(s))dw(s). \tag{65}$$

Let $(\mathbb{C}[0,T], C_t), t \in [0,T]$ be a space of continuous functions on the interval $[0,T]$. Let μ_ε and μ denote the measures on functional space $(\mathbb{C}[0,T], C_t)$ generated by the processes $\xi_\varepsilon(t)$ and $\xi(t)$, respectively.

Theorem 1. *We establish in [5, 6] the necessary and sufficient conditions for the weak convergence of solutions of stochastic equations (1) to the solution of stochastic equation (2)*

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R E F E R E N C E S

1. Harrison, J.M., Shepp, L.A. *On skew Brownian motion*, Ann. Probab. **9**(2), 309-313 (1981).

2. Le Gall, J.-F. *One dimensional stochastic differential equations involving the local times of the unknown process*, Lecture Notes in Math. **1095**, 51-82 (1983).

3. Engelbert, H.J., Schmidt W. *Strong Markov continuous local martingales and solutions of one-dimensional stochastic differential equations III*, Math. Nachr. **151**, 149-197 (1991).

4. Makhno, S.Ya. *A limit theorem for stochastic equations with the local time*, Theory of Prob. and Math. Statistics **64**, 123-128 (2002).

5. Krykun, I.H. *On weak convergence of stochastic diferential equations with irregular coefcients*, Journal of Mathematical Sciences **273**(3), 398-413 (2023).

6. Krykun, I.H. *About convergence of solutions of one-dimensional stochastic equations*, Stochastics and Dynamics **26**(5), (2026).

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БАКСТЕРІВСЬКА ОЦІНКА ПАРАМЕТРА МАСШТАБУ ОДНОГО ВИПАДКОВОГО ПРОЦЕСУ МАТЕРНА

П. Леві [1] довів, що для стандартного броунівського руху $W(t), t \in [0,1]$, має місце збіжність $\sum_{k=1}^{2^n}\left(W\left(\frac{k}{2^n}\right)-W\left(\frac{k-1}{2^n}\right)\right)^2 \to 1, n \to \infty$ з ймовірністю 1. Пізніше, Г.Бакстер [2] узагальнив цей результат для певного класу гауссових випадкових процесів. Такі граничні теореми називають бакстерівського типу, а відповідні суми — бакстерівськими [3].

Бакстерівські теореми застосовують у статистиці параметричного оцінювання. Так, у статті [4] за допомогою бакстерівських статистик отримано сильно консистентну оцінка параметра Хюрста дробового броунівського руху. У монографії [3] бакстерівські суми застосовували при оцінюванні параметрів випадкових функцій у різних статистичних моделях.

Випадковий процес Матерна $X = \{X(t), t \in \mathbb{R}\}$ — це центрований стаціонарний гауссівський процес з $\operatorname{Var} X(t) = \sigma^2$ і коваріаційною функцією:

$$k_{Matern}(t) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)}\left(\frac{\sqrt{2\nu}|t|}{l}\right)^{\nu} K_\nu\left(\frac{\sqrt{2\nu}|t|}{l}\right), \quad t \in \mathbb{R}\setminus\{0\},$$

із додатніми параметрами масштабу l та гладкості ν, $\Gamma(\cdot)$ — гамма-функція, а $K_\nu(\cdot)$ — модифікована функція Бесселя 2-го роду порядку ν, $k_{Matern}(0)$
$= \sigma^2$ [5]. При $\nu = \frac{3}{2}$ коваріаційна функція Матерна має вигляд:

$$k_{\sigma^2,l}(t) = \sigma^2\left(1+\frac{\sqrt{3}}{l}|t|\right)\exp\left\{-\frac{\sqrt{3}}{l}|t|\right\}. \tag{66}$$

Зауважимо, що коваріацінa функція (66) є нескінченно диференційовною на $\mathbb{R}\setminus\{0\}$ і двічі неперервно диференційовною на всій множині $\mathbb{R}$.

Випадквоий процес Матерна з коваріаційною функцією (66) застосовується у моделях просторової статистики, геостатистики, а також машинному навчанні для побудови моделей гауссівських процесів, просторового прогнозування, інтерполяції та згладжування даних.

Для спрощення запису покладемо $\alpha = \frac{\sqrt{3}}{l}, \beta = \frac{\alpha^3}{3}$ і будемо оцінювати параметр β.

Постановка задачі оцінювання. Побудувати оцінку параметра β випадкового процесу Матерна з коваріаційною функцією (66) при $\sigma^2 = 1$ за спостереженнями у точках

$$0, \frac{0,5}{a_n}, \frac{1}{a_n}, \frac{1.5}{a_n}, ..., \frac{a_n-1}{a_n}, \frac{a_n-0.5}{a_n}, 1,$$

де $(a_n) \in \mathbb{N}, a_n \to \infty, n \to \infty$.

Для розв'язування цієї задачі застосуємо бакстерівські суми з приростами 2-го порядку:

$$S_n^{(2)} = \sum_{k=0}^{a_n-1}\left(X\left(\frac{k+1}{a_n}\right)-2X\left(\frac{k+0.5}{a_n}\right)+X\left(\frac{k}{a_n}\right)\right)^2.$$

Статистика $\hat{\beta}_n = a_n^2 S_n^{(2)}, n \geq 1$ має такі властивості:

$$\begin{aligned} &1.\ \ \mathrm{E}\,\hat{\beta}_n \to \beta, n \to \infty; \\ &2.\ \ \mathrm{Var}\,\hat{\beta}_n = O\left(\frac{1}{a_n}\right), n \to \infty. \end{aligned} \tag{67}$$

Теорема 1. *Статистика $\hat{\beta}_n$ є консистентною оцінкою параметра β. Якщо $\sum_{n=1}^{\infty} \frac{1}{a_n} < +\infty$, то $\hat{\beta}_n$ — сильно консистентна оцінка параметра β.*

Побудуємо довірчий інтервал з рівнем довіри $1-p$ для параметра β. Нехай $\varepsilon > 0$. Внаслідок (67),

$$\exists N \in \mathbb{N} \quad \forall n \geq N : |\,\mathrm{E}\,\hat{\beta}_n - \beta| < \frac{\varepsilon}{2}. \tag{68}$$

Теорема 2. *Нехай N визначено у (68). Покладемо*

$$\overline{n} = \min\left\{N \in \mathbb{N} \mid n \geq N \quad i \quad \frac{4\,\mathrm{Var}\,\hat{\beta}_n}{\varepsilon^2} < p\right\}.$$

Тоді $P\left\{\beta \in \left(C_{\overline{n}}, \hat{\beta}_{\overline{n}} + \varepsilon\right)\right\} > 1 - p$, де $C_{\overline{n}} = \max\{0, \hat{\beta}_{\overline{n}} - \varepsilon\}$.

Л І Т Е Р А Т У Р А

1. Levy, P. *Le movement Brownian plan*, Amer. J. Math. **62**, 487– 550 (1940).

2. Baxter, G. *A strong limit theorem for Gaussian processes*, Proc. Amer. Math. Soc. **7**(3), 522–527 (1956).

3. Козаченко, Ю.В., Курченко, О.О., Синявська, О.О. Теореми Леві-Бакстера для випадкових полів та їх застосування: Монографія. Ужгород (2018).

4. Курченко, О.О. *Одна сильно консистентна оцінка параметра Хюрста дробового броунівського руху*, Теор. ймов. та матем. статистика. **67**, 45–54 (2002).

5. Rasmussen, C.E., Williams C.K.I. Gaussian Processes for Machine Learning, the MIT Press, 2006.

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DIFFERENTIABLE UPPER BOUND FUNCTION FRAMEWORK FOR SURROGATE MIXING PROBABILITIES CONSTRUCTION IN A MODEL OF MIXTURE OF VARYING CONCENTRATIONS

We consider a parameter estimation problem by observation from a mixture with a finite number of components. Finite mixture models have a deep history, part of which can be read in the works [1] and [2]. Mixture models are used for sociological, medical and biological data description. There exist parametric and non-parametric models of mixtures. In study [3] a non-parametric minimax approach was applied to a finite mixture model with applications to DNA microarray data analysis.

We assume that each observed subject O belongs to one of M different mixture components ρ_m, $m = \overline{1,M}$. The sample contains n subjects $O_1, ..., O_n$. Let $\kappa_j = m$ iif $O_j \in \rho_m$. The true κ_j are unknown, but one knowns the mixing probabilities

$$p_j^m = p_{j;n}^m = P(\kappa_j = m).$$

These mixing probabilities are used to to build minimax weights $A_n = (a_j^m)_{j=\overline{1,n}}^{m=\overline{1,M}} = P_n(P_n^T P_n)^{-1}$, where $P_n = (p_j^m)_{j=\overline{1,n}}^{m=\overline{1,M}}$ and these weights are used to build various statistical models like regression models.

The D-dimensional vector of observed variables of O will be denoted by $\xi(O) = (\xi^1(O), ..., \xi^D(O))^T$, $\xi_j = \xi(O_j)$.

Let $F^{(m)}$ be distribution of $\xi(O)$ for $O \in \rho_m$,

$$F^{(m)}(A) = P\{\xi(O) \in A | O \in \rho_m\}, \text{ and } P\{\xi_j \in A\} = \sum_{m=1}^{M} p_j^m F^{(m)}(A).$$

for all Borel $A \subset R^D$.

In many problems, mixing probabilities are not observed, but they can be replaced by surrogates. In this work we are building such kind of mixing probabilities surrogate and assume their dependency on vector of parameters α. So,

$$\Big(a_j^i(\alpha)\Big)_{j=\overline{1,n}}^{i=\overline{1,M}} = A_n(\alpha) = P_n(\alpha)\Big(P_n(\alpha)^T P_n(\alpha)\Big)^{-1},$$

where $P_n(\alpha) = (p_j^i(\alpha))_{j=\overline{1,n}}^{i=\overline{1,M}}$ is a mixing probability matrix. The value $p_j^i(\alpha)$ represents the probability that the j-th element belongs to the component i given the parameters α. This vector α should be estimated.

In this talk we consider a differentiable upper bound function framework for the estimation of the parameter α and demonstrate it on classic and well known problem of mean value estimation with MVC [4] and KNN clustering [5, p. 509]. Let's remind a KNN clustering loss function, where M is a fixed number of clusters, and S_k is a set of cluster k and elements x are from sample $\Xi_n = (\xi_1, ..., \xi_n)$: $Loss_{knn} = \sum_{k=1}^{M} \sum_{x \in S_k} ||x - \mu_k||^2$.

KNN clusterizer is fit by minimizing $Loss_{knn}$ over all sample splits into M clusters. Usually μ_k, $1 \le k \le n$ is mean value inside cluster S_k.

Now we use the MVC model and replace $Loss_{knn}$ by its analogue. The idea is to fit mixing probabilities parameters by minimizing this loss function:

$$Loss_a(\alpha,\mu) = \sum_{k=1}^{M}\sum_{i=1}^{n} a_j^i(\alpha)||\xi_i - \mu_k||^2.$$

This function has no lower bound and might obtain any negative value. That is why we consider a different loss function:

$$Loss_{bound}(\alpha,\mu) = \sqrt{\sum_{k=1}^{M}\frac{1}{\lambda_k^2(\alpha)}}\sqrt{\sum_{k=1}^{M}(\sum_{i=1}^{n} p_i^k(\alpha)||\xi_i-\mu_k||^2)^2},$$

where $\lambda_k(\alpha)$ are eigenvalues of the matrix $P_n(\alpha)^T P_n(\alpha)$.

It can be shown that $|Loss_a(\alpha,\mu)| \le Loss_{bound}(\alpha,\mu)$,

and minimization problem for $Loss_{bound}(\alpha,\mu)$ is much easier, especially for regression models with a large number of unknown parameters α inside mixing function $P_n(\alpha)$.

As a result, we will have following parameter estimators: $\hat{\mu}_k(\alpha) = \sum_{j=1}^{n} a_j^k(\alpha)\xi_j$, $\hat{\mu}(\alpha) = (\hat{\mu}_1(\alpha), ..., \hat{\mu}_M(\alpha))$ and surrogate mixing probabilities estimator for the parameter α is defined as

$$\hat{\alpha} =_{\alpha} Loss_{bound}(\alpha, \hat{\mu}(\alpha))$$

REFERENCES

1. Hall P., Titterington D., *The Use of Uncategorized Data to Improve the Performance of a Nonparametric Estimator of a Mixture Density*, Journal of the Royal Statistical Society. Series B (Methodological), **Vol. 47** (1985), No. 1, pp. 155-163

2. Mclachlan G., Peel D., *Finite mixture models*, Wiley Series in Probability and statistics, John Wiley, New York, 2000

3. Maiboroda R., Sugakova O., *Statistics of mixtures with varying concentrations with application to DNA microarray data analysis*, Journal of nonparametric statistics **Vol. 24** (2012), No. 1, pp. 201-205

4. Maiboroda R., Kubaichuk O., *Improved Estimators for Moments Constructed form Observations of a Mixture*, Theory of Probability and Mathematical Statistics **70** (2005), pp. 83-92, doi:10.1090/S0094-9000-05-00642-3.

5. Hastie T., Tibshirani R., Friedman J., *The Elements of Statistical Learning: Data Mining, Inference, and Prediction.* 12nd ed, Springer, 2017, 509 pp.

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DRIFT ESTIMATION IN THE CKLS MODEL: ASYMPTOTIC NORMALITY AND SIMULATIONS

We consider the Chan–Karolyi–Longstaff–Sanders model

$$\mathrm{d}r_t = (a - br_t)\,\mathrm{d}t + \sigma r_t^{\beta}\,\mathrm{d}W_t, \qquad a, b, \sigma > 0, \quad \beta \in (1/2, 1), \quad r_0 > 0, \tag{69}$$

where W is a Wiener process. This model was introduced in [1] for short-term interest-rate dynamics and contains the Cox–Ingersoll–Ross model as the limiting case $\beta = 1/2$.

Assume that the trajectory $\{r_t, t \in [0, T]\}$ is observed continuously and that σ and β are known. Put

$$I_\alpha(T) = \int_0^T r_t^\alpha\,\mathrm{d}t, \qquad D_T = \frac{I_{3-2\beta}(T)}{T} - \frac{I_1(T) I_{2-2\beta}(T)}{T^2}.$$

We study the strongly consistent estimators introduced in [2]:

$$\widetilde{a}_T = \frac{\sigma^2(1-\beta) I_1^2(T)}{T^2 D_T}, \qquad \widetilde{b}_T = \frac{\sigma^2(1-\beta) I_1(T)}{T D_T}. \tag{70}$$

The process is strictly positive and ergodic. If p_∞ denotes its invariant density and

$$\mu_\alpha = \int_0^\infty x^\alpha p_\infty(x)\,\mathrm{d}x,$$

then $\mu_\alpha < \infty$ for every $\alpha \in \mathbb{R}$ and

$$\frac{1}{T} I_\alpha(T) \longrightarrow \mu_\alpha \quad \text{a.s.}$$

We also prove the uniform moment estimate

$$\sup_{t \ge 0} \mathbb{E} r_t^p < \infty, \qquad p > 0,$$

which extends the second-moment bound established in [3].

Theorem 1. *As $T \to \infty$,*

$$\sqrt{T} \begin{pmatrix} \widetilde{a}_T - a \\ \widetilde{b}_T - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma),$$

where

$$\Sigma = \frac{b^4}{a^2\sigma^2(1-\beta)^2} A \Gamma A^\top, \quad A = \begin{pmatrix} \mu_{3-2\beta} & a/b \\ \mu_{2-2\beta} & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \mu_{2\beta} & -\mu_2 \\ -\mu_2 & \mu_{4-2\beta} \end{pmatrix}.$$

The proof is based on a representation of the normalized estimation error as a linear transformation of the two-dimensional Itô martingale

$$Z_T = T^{-1/2} \begin{pmatrix} \int_0^T r_t^\beta\,\mathrm{d}W_t \\ -\int_0^T r_t^{2-\beta}\,\mathrm{d}W_t \end{pmatrix}.$$

Its normalized quadratic variation converges to Γ by ergodicity; therefore, the multidimensional martingale central limit theorem gives $Z_T \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Gamma)$. The uniform moment estimate makes the boundary remainders negligible, and Slutsky's theorem yields Theorem 1.

Numerical experiment. We used $a = 1$, $b = 2$, $\sigma = 0.5$, $r_0 = 1$, $\beta \in \{0.55, 0.70, 0.90\}$, and $T \in \{100, 300, 1000\}$. For every pair (β, T), $N = 1000$ independent trajectories were generated by a truncated Euler scheme with $\Delta = 10^{-3}$, and the integrals in (70) were approximated by left Riemann sums.

β	T	$\bar{a}_T$	$\bar{b}_T$	RMSE_a	RMSE_b	mean(Q_T)	coverage
0.55	100	1.0028	1.9958	0.1035	0.2178	2.0255	0.943
0.55	300	1.0010	1.9979	0.0613	0.1275	2.0403	0.946
0.55	1000	0.9994	1.9992	0.0336	0.0721	2.0662	0.945
0.70	100	1.0022	1.9928	0.1083	0.2264	2.0661	0.944
0.70	300	1.0003	1.9976	0.0629	0.1350	2.0367	0.948
0.70	1000	1.0006	2.0009	0.0330	0.0710	1.9057	0.954
0.90	100	1.0029	1.9982	0.1057	0.2295	1.8915	0.960
0.90	300	0.9977	1.9918	0.0619	0.1352	1.9482	0.953
0.90	1000	1.0002	1.9994	0.0353	0.0749	1.9815	0.958

Here

$$Q_T = T \begin{pmatrix} \widetilde{a}_T - a \\ \widetilde{b}_T - b \end{pmatrix}^{\top} \Sigma^{-1} \begin{pmatrix} \widetilde{a}_T - a \\ \widetilde{b}_T - b \end{pmatrix}, \qquad Q_T \xrightarrow{d} \chi^2_2.$$

The empirical means are close to the true values $a = 1$ and $b = 2$, while the RMSEs decrease as T grows. The scaled RMSEs are stable: $\sqrt{T}\,\mathrm{RMSE}(\widetilde{a}_T)$ lies between 1.035 and 1.115, and $\sqrt{T}\,\mathrm{RMSE}(\widetilde{b}_T)$ lies between 2.178 and 2.368, supporting the $T^{-1/2}$ convergence rate.

The empirical variances differ from the corresponding theoretical entries of Σ by at most 8.5% for $\widetilde{a}_T$ and 7.4% for $\widetilde{b}_T$. The maximal absolute difference between empirical and theoretical correlations is 0.0065. Since $\mathbb{E}\chi^2_2 = 2$ and $\mathrm{Var}(\chi^2_2) = 4$, the observed values mean(Q_T) $\in [1.892, 2.066]$, Var(Q_T) $\in [3.388, 4.994]$, and coverage 0.943–0.960 support the bivariate normal approximation. The Q–Q plots agree well with χ^2_2 in the central region, with moderate finite-sample deviations in the upper tail.

Thus, the simulations illustrate the small bias, the $T^{-1/2}$ rate, the covariance matrix Σ, and the asymptotic χ^2_2 behavior implied by Theorem 1.

REFERENCES

1. Chan, K.C., Karolyi, G.A., Longstaff, F.A., Sanders, A.B. *An empirical comparison of alternative models of the short-term interest rate*, J. Finance **47**, 1209–1227 (1992).

2. Mishura, Y., Ralchenko, K., Dehtiar, O. *Parameter estimation in CKLS model by continuous observations*, Statist. Probab. Lett. **184**, Paper No. 109391 (2022).

3. Mishura, Y., Pilipenko, A., Ralchenko, K. *Gatheral double stochastic volatility model with Skorokhod reflection*, Theory Probab. Math. Statist. **113**, 153–171 (2025).

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TEMPERED FRACTIONAL ORNSTEIN–UHLENBECK PROCESSES: ASYMPTOTICS AND INFERENCE

Tempered fractional Brownian motion preserves the local fractional behavior of fractional Brownian motion, while exponential tempering weakens dependence at large lags. We study stationary and non-stationary Ornstein–Uhlenbeck processes driven by its variants of the first and second kind, extending the classical fractional model [3]. Our aims are to derive covariance asymptotics, establish ergodicity, and estimate the drift parameter from continuous- and discrete-time observations.

Let $H > 0$, $\lambda > 0$, $\theta > 0$, and let $W = \{W_x, x \in \mathbb{R}\}$ be a two-sided Wiener process. For $k \in \{\mathrm{I}, \mathrm{II}\}$, let $B^k_{H,\lambda}$ denote tempered fractional Brownian motion of the first kind [1] or the second kind [2], respectively. The processes under consideration solve $dX^k_t = -\theta X^k_t\, dt + dB^k_{H,\lambda}(t)$. The stationary solution is

$$Y^k_t = B^k_{H,\lambda}(t) - \theta e^{-\theta t} \int_{-\infty}^{t} e^{\theta s} B^k_{H,\lambda}(s)\, ds,$$

whereas the non-stationary solution with $X^k_0 = 0$ is given by

$$X^k_t = B^k_{H,\lambda}(t) - \theta e^{-\theta t} \int_{0}^{t} e^{\theta s} B^k_{H,\lambda}(s)\, ds.$$

The process Y^k admits the Wiener integral representation $Y^k_t = \int_{\mathbb{R}} h^k(t-x)\, dW_x$, where

$$h^k(z) = \left(e^{-\lambda z} z^{H-1/2} - a_k e^{-\theta z} \int_0^z e^{(\theta-\lambda)y} y^{H-1/2}\, dy \right) \mathbf{1}_{(0,\infty)}(z),$$

$a_{\mathrm{I}} = \theta$ and $a_{\mathrm{II}} = \theta - \lambda$. Thus, Y^k is a centered stationary Gaussian process with autocovariance

$$r_k(\tau) = \int_{\mathbb{R}} h^k(\tau+u) h^k(u)\, du.$$

Theorem 1. *As $\tau \to \infty$, the autocovariance functions satisfy the following asymptotic relations:*

	$r_{\mathrm{I}}(\tau)$	$r_{\mathrm{II}}(\tau)$
$\theta < \lambda$	$\sim -\dfrac{\theta\Gamma(H+\frac12)^2}{2(\lambda^2-\theta^2)^{H+\frac12}} e^{-\theta\tau}$	$\sim \dfrac{\Gamma(H+\frac12)^2}{2\theta(\lambda^2-\theta^2)^{H-\frac12}} e^{-\theta\tau}$
$\theta = \lambda$	$\sim -\dfrac{\Gamma(H+\frac12)}{(H+\frac12)2^{H+\frac32}\theta^{H-\frac12}} e^{-\theta\tau}\tau^{H+\frac12}$	$\sim \dfrac{\Gamma(H+\frac12)}{2^{H+\frac12}\theta^{H+\frac12}} e^{-\theta\tau}\tau^{H-\frac12}$
$\theta > \lambda$	$\sim -\dfrac{\Gamma(H+\frac12)}{2^{H+\frac12}\lambda^{H-\frac32}(\theta^2-\lambda^2)} e^{-\lambda\tau}\tau^{H-\frac12}$	$\sim \dfrac{(H-\frac12)\Gamma(H+\frac12)}{2^{H-\frac12}\lambda^{H-\frac12}(\theta-\lambda)(\theta+\lambda)} e^{-\lambda\tau}\tau^{H-\frac32},\ H \ne \dfrac12$

In the exceptional case $k = \mathrm{II}$, $H = 1/2$, $\theta > \lambda$, one has $r_{\mathrm{II}}(\tau) = (2\theta)^{-1} e^{-\theta\tau}$ for $\tau \ge 0$. Consequently, $r_k(\tau) \to 0$ and Y^k is ergodic.

For drift estimation, assume that H and λ are known and set $\psi_k(\theta) = \mathbb{E}_\theta (Y^k_0)^2$. Following the ergodic-type approach of [4, 5], we obtain

$$\psi_{\mathrm{I}}(\theta) = \frac{\Gamma(H+1/2)^2}{\pi} \int_0^\infty \frac{\omega^2\, d\omega}{(\theta^2+\omega^2)(\lambda^2+\omega^2)^{H+1/2}},$$

$$\psi_{\mathrm{II}}(\theta) = \frac{\Gamma(H+1/2)^2}{\pi} \int_0^\infty \frac{d\omega}{(\theta^2+\omega^2)(\lambda^2+\omega^2)^{H-1/2}}.$$

Both functions are continuous and strictly decreasing on $(0,\infty)$, so θ is identifiable. For every fixed $h>0$,

$$\frac{1}{T}\int_0^T (X_t^k)^2\,dt \to \psi_k(\theta), \qquad \frac{1}{N}\sum_{j=1}^N (X_{jh}^k)^2 \to \psi_k(\theta) \quad \text{a.s.}$$

This motivates the estimators

$$\widehat{\theta}_T^k = \psi_k^{-1}\left(\frac{1}{T}\int_0^T (X_t^k)^2\,dt\right), \qquad \widehat{\theta}_{N,h}^k = \psi_k^{-1}\left(\frac{1}{N}\sum_{j=1}^N (X_{jh}^k)^2\right).$$

For $k=\mathrm{I}$, they are well defined for all sufficiently large T and N, almost surely.

Theorem 2. *The estimators are strongly consistent. Moreover, as $T\to\infty$ and, for every fixed $h>0$, as $N\to\infty$,*

$$\sqrt{T}\left(\widehat{\theta}_T^k-\theta\right) \xrightarrow{d} \mathcal{N}\left(0,\frac{\sigma_k^2}{(\psi_k'(\theta))^2}\right), \qquad \sqrt{N}\left(\widehat{\theta}_{N,h}^k-\theta\right) \xrightarrow{d} \mathcal{N}\left(0,\frac{\sigma_{k,h}^2}{(\psi_k'(\theta))^2}\right),$$

where

$$\sigma_k^2 = 2\int_{\mathbb{R}} r_k(\tau)^2\,d\tau, \qquad \sigma_{k,h}^2 = 2\sum_{m\in\mathbb{Z}} r_k(mh)^2.$$

Thus, tempering yields the standard square-root normalization for every $H>0$, unlike the classical fractional model [3].

R E F E R E N C E S

1. Meerschaert, M.M., Sabzikar, F. *Tempered fractional Brownian motion*, Statist. Probab. Lett. **83**, 2269–2275 (2013).

2. Sabzikar, F., Surgailis, D. *Tempered fractional Brownian and stable motions of second kind*, Statist. Probab. Lett. **132**, 17–27 (2018).

3. Cheridito, P., Kawaguchi, H., Maejima, M. *Fractional Ornstein–Uhlenbeck processes*, Electron. J. Probab. **8**, paper no. 3, 1–14 (2003).

4. Hu, Y., Nualart, D. *Parameter estimation for fractional Ornstein–Uhlenbeck processes*, Statist. Probab. Lett. **80**, 1030–1038 (2010).

5. Hu, Y., Nualart, D., Zhou, H. *Parameter estimation for fractional Ornstein–Uhlenbeck processes of general Hurst parameter*, Stat. Inference Stoch. Process. **22**, 111–142 (2019).

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ON COMPONENT DOMINANCE TRANSITION IN DYNAMIC MIXTURES WITH SIGMOIDAL WEIGHTS

Classical finite mixtures of exponential distributions with constant weights are well studied and widely applied for modeling heterogeneous populations [1]. However, when modeling phenomena with sharp transitional regimes, such models lack flexibility. In this context, dynamic (state-dependent) mixtures have been introduced, in which the mixing weights depend on the state variable x or additional covariates [2]. Extending the framework of sigmoidal weights [3], we consider the mixture density (PDF) $p(x)$ defined for $x \geq 0$:

$$p(x) = \frac{1}{C}\left(w(x;a,r)\lambda e^{-\lambda x} + (1 - w(x;a,r))\mu e^{-\mu x}\right),$$

where $\lambda, \mu > 0$ are the component rate parameters, $a > 0$ is the transition threshold, and the logistic weight function

$$w(x;a,r) = \sigma(r(x-a)) = \frac{1}{1+e^{-r(x-a)}}, \qquad r > 0,$$

controls the transition of component dominance along $x \geq 0$.

This study derives an explicit closed-form expression for the normalization constant C in terms of the Lerch Phi function:

$$C = \frac{\lambda}{r}\Phi\left(-e^{ra}, 1, \frac{\lambda}{r}\right) + 1 - \frac{\mu}{r}\Phi\left(-e^{ra}, 1, \frac{\mu}{r}\right).$$

The continuous probability density $p(x)$ is annihilated by the linear differential operator $\mathcal{L}$:

$$\mathcal{L}p(x) \equiv p''(x) + P(x;a,r)p'(x) + Q(x;a,r)p(x) = 0,$$

where the coefficients of the annihilating operator are explicitly given by:

$$\begin{aligned} P(x;a,r) &= 2rw(x;a,r) - r + \lambda + \mu, \\ Q(x;a,r) &= (r-\lambda)\mu - r(\lambda+\mu)w(x;a,r). \end{aligned}$$

To analyze the limiting behavior as $r \to \infty$, we consider the unnormalized density $p^*(x) = C \cdot p(x)$. Applying the Method of Matched Asymptotic Expansions (MAE) with Van Dyke's matching principle [4], the outer expansions outside the boundary layer ($|x-a| \gg 1/r$) reduce to:

$$\begin{aligned} p^-_{out}(x) &= g(x) = \mu e^{-\mu x}, \qquad & 0 \leq x < a, \\ p^+_{out}(x) &= f(x) = \lambda e^{-\lambda x}, \qquad & x > a. \end{aligned}$$

Inside the transition layer of thickness $\mathcal{O}(1/r)$, accounting for the difference $(f(a) - g(a))$, Van Dyke's procedure yields the composite asymptotic density:

$$p^*_{comp}(x) = \begin{cases} g(x) + (f(a) - g(a))\sigma(r(x-a)), & 0 \leq x \leq a, \\ f(x) + (f(a) - g(a))(\sigma(r(x-a)) - 1), & x > a. \end{cases}$$

The uniform error estimate,

$$\|p^*(x) - p^*_{comp}(x)\|_{C[0,\infty)} = \mathcal{O}(1/r),$$

guarantees first-order asymptotic accuracy over the entire domain $x \in [0, \infty)$.

Conclusions. This study characterizes the transition of component dominance in dynamic mixture distributions. Constructing the annihilating differential operator and applying Van Dyke's matching principle show that the sigmoidal weight induces an internal transition layer of width $\mathcal{O}(1/r)$ around the threshold a. The constructed composite density $p^*_{comp}(x)$ guarantees a uniformly valid $\mathcal{O}(1/r)$-approximation across $x \geq 0$.

R E F E R E N C E S

1. Cossette, H., Marceau, E., Mtalai, I., Veilleux, D. *Univariate and multivariate mixtures of exponential distributions, with applications in risk modeling*, Appl. Stoch. Models Bus. Ind. **37**(4), 675-702 (2021).

2. Bee, M. *A parsimonious dynamic mixture for heavy-tailed distributions*, Math. Comput. Simul. **230**, 193-206 (2025).

3. Tuluchenko, H.Ya. *Dynamically Weighted Mixture of Exponential and Erlang Distributions with Independent Rate Parameters*, XX International Scientific Conference Named After Academician Mykhailo Kravchuk, Igor Sikorsky Kyiv Polytechnic Institute, Kyiv, 143-144 (2025).

4. Holmes, M.H. *Introduction to Perturbation Methods*, Springer, New York (2013).

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THE ARCTAN-TAN-G DISTRIBUTION FAMILY: PROPERTIES AND APPLICATIONS

Nowadays there exist many distribution families which are obtained by a trigonometric transformation of a base distribution. Such families include, in particular, the Sin-G ([2]), the Tan-G ([3]), the Arctan-X ([1]) models, to name but a few.

A new family of distributions, the Arctan-Tan-G family, is proposed. The cdf of this model is defined as

$$F(x) = \frac{1}{\arctan a} \cdot \arctan\left(a \tan\left(\frac{\pi}{4} G(x)\right)\right), \tag{71}$$

where $a > 0$ and $G(x)$ is the cdf of a base distribution.

Various properties of the new family are studied, including moments, quantile-based moments, order statistics and stochastic ordering. A member of the Arctan-Tan-G family, the Arctan-Tan-Weibull distribution, is considered and its properties are obtained. Suitability of the Arctan-Tan-Weibull distribution for modeling real data is verified by fitting this distribution to a survival data set.

R E F E R E N C E S

1. Alkhairy, I., Nagy, M., Muse, A.H., Hussam, E. *The Arctan-X family of distributions: properties, simulation, and applications to actuarial sciences*, Complexity **2021** (1), 4689010 (2021).

2. Kumar, D., Kumar, S., Singh, U. *A new distribution using sine function — its application to bladder cancer patients data*, J. Stat. Applic. Probab. **4** (3), 417-427 (2015).

3. Souza, L., O. de Júnior, W. R. D., de Brito, C. C. R., Chesneau, C., Fernandes, R. L., & Ferreira, T. A. E. *Tan-G class of trigonometric distributions and its applications*, CUBO **23** (1), 1-20 (2021).

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LIMIT THEOREMS FOR KARLIN'S OCCUPANCY SCHEME

Karlin's occupancy scheme is defined as follows. Balls are thrown independently into an infinite array of boxes numbered $1, 2, \ldots$ with probability p_k of hitting box k. Here, $(p_k)_{k\geq 1}$ is a discrete probability distribution with infinitely many $p_k > 0$.

In practice, the role of boxes is played by the types or species of sample items. Such a model with infinitely many boxes can approximate a sample from a large finite population. As examples of possible applications, I will consider the risk estimation of personal information disclosure and literature analysis.

The primary quantity of interest is the number of occupied boxes, denoted by K_n, after n balls have been distributed. First, I will discuss the common assumptions imposed on the distribution $(p_k)_{k\geq 1}$.

Then, I will provide an overview of the limit theorems for the number of occupied boxes K_n, including recent laws of the iterated and single logarithm [1].

R E F E R E N C E S

1. D. Buraczewski, A. Iksanov and V. Kotelnikova, *Laws of the iterated and single logarithm for sums of independent indicators, with applications to the Ginibre point process and Karlin's occupancy scheme.* Stochastic Process. Appl. **183** (2025), Paper No. 104597.

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HADAMARD FRACTIONAL BROWNIAN MOTION AND OTHER GAUSSIAN VOLTERRA PROCESSES

This is a common talk with L.Beghin and K. Ralchenko. We consider various types of Gaussian processes with Volterra kernels. In particular, we study the Hadamard fractional Brownian motion that is a Gaussian process which shares some properties with standard Brownian motion (such as the one-dimensional distribution). However, it also resembles the fractional Brownian motion in many other features as, for instance, selfsimilarity, long/short memory property, Wiener-integral representation. The logarithmic kernel in the Hadamard fractional Brownian motion represents a very specific and interesting aspect of this process. Our aim here is to analyze some properties of the process' trajectories (i.e. Hölder continuity, quasi-helix behavior, power variation, local nondeterminism) that are both interesting on their own and serve as a basis for the Wiener integration with respect to it. The respective integration is quite well developed, and the inverse representation is also constructed. We apply the derived “multiplicative Sonine pairs” to the treatment of the Reproducing Kernel Hilbert Space of the Hadamard fractional Brownian motion, and, as a result, we establish a law of iterated logarithm.

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MATHEMATICAL MODELING OF DYNAMIC PROCESSES IN COMPUTER SYSTEMS

Modern computer systems are characterized by highly dynamicc operation, a large number of interconnected components, and irregular patterns of information and computational request arrivals. The increasing volume of data, number of users, and complexity of information processes necessitate the development of adequate mathematical models capable of describing the behavior of computer systems under different load conditions. Of particular importance is the modeling of processes in which system characteristics vary over time, while request arrival times and processing durations are stochastic in nature [2]. One of the most widely used approaches to the mathematical modeling of dynamic processes in computer systems is to represent them as queueing systems. This is due to the fact that a significant proportion of computer system operations can be described in terms of a stochastic stream of requests arriving at a limited number of service resources.

The aim of this work is to investigate a mathematical model of dynamic processes in computer systems based on stochastic queueing systems. To achieve this aim, the main operational parameters of a computer system are formalized in terms of a queueing system, the dependence of its characteristics on the parameters of the input stream and service process is investigated, and the main patterns of variation in system performance indicators over time are identified. To describe the system dynamics, let us introduce the state probabilities $P_n(t) = P\{N(t) = n\}$, where $P_n(t)$ is the probability that there are n requests in the system at time t. Obviously, the following normalization condition holds at any time: $\sum_{n=0}^{K} P_n(t) = 1$. For the internal states $1 \le n < K$, the dynamics of the probabilities are described by the system of Kolmogorov differential equations [1; 3]:

$$\frac{dP_n(t)}{dt} = \lambda(t)P_{n-1}(t) + \mu_{n+1}(t)P_{n+1}(t) - [\lambda(t) + \mu_n(t)]P_n(t),$$

where $\mu_n(t) = \min(n, c)\mu(t)$ is the total service rate in state n.

Here:

- $\lambda(t)P_{n-1}(t)$ represents the transition from state $n-1$ to state n due to the arrival of a new request;
- $\mu_{n+1}(t)P_{n+1}(t)$ represents the transition from state $n+1$ to state n due to the completion of service;
- $[\lambda(t) + \mu_n(t)]P_n(t)$ is the total rate of leaving state n.

For the boundary state $n = K$, the equation is

$$\frac{dP_K(t)}{dt} = \lambda(t)P_{K-1}(t) - \mu_K(t)P_K(t).$$

For a system with finite capacity K [1; 3]: $\mu_K(t) = \min(K, c)\mu(t)$.

The following indicators are used for the quantitative assessment of the dynamics of a computer system.

The average number of requests in the system is $L(t) = \sum_{n=0}^{K} nP_n(t)$.

The average number of requests in the queue is $L_q(t) = \sum_{n=c+1}^{K}(n - c)P_n(t)$.

The average number of busy service channels is $B(t) = \sum_{n=0}^{K} \min(n, c)P_n(t)$.

The utilization coefficient of the service resources can then be determined as

$$U(t) = \frac{B(t)}{c}.$$

The obtained relationships make it possible to investigate not only the current state of the system but also the nature of its variation over time. In particular, it is possible to determine the onset of overload, the duration of the transient process, the rate of queue accumulation, and the time required for the system to return to an acceptable operating mode. For a practical study, the initial state of the system must also be specified. Since most modern computer systems are multitasking, let us consider a multichannel queueing system in which several service devices operate simultaneously, namely an $M/M/3$ system. For $c = 3$ and $\mu = 5$ requests/s, the total service capacity is $c \cdot \mu = 15$ requests/s. Therefore, $\rho(t) = \frac{\lambda(t)}{15}$. For the specified operating modes:

$$\begin{aligned} &\lambda_1 = 6, \quad \rho_1 = 0.4, \quad \lambda_2 = 12, \quad \rho_2 = 0.8, \\ &\lambda_3 = 18, \quad \rho_3 = 1.2, \quad \lambda_4 = 8, \quad \rho_4 = \frac{8}{15} \approx 0.533. \end{aligned}$$

Thus, the mathematical model makes it possible to formally describe the transition of the system through four operating modes:

$$0.4 \to 0.8 \to 1.2 \to 0.533.$$

This corresponds to the transition of the system from stable operation to high load, overload, and subsequent recovery.

R E F E R E N C E S

1. Kvietnyi R. N., Bogach I. V., Boiko O. R., Sofina O. Yu., and Shushura O. M., *Computer Modeling of Systems and Processes. Computational Methods. Part 1.* Vinnytsia: VNTU, 2012, 193 p.

2. Kolomiiets T., Mykhailenko V., Pohorui A., Postova S., Franovskyi A., *Mathematical Modeling of Dynamic Systems and Processes: Instructional and Methodological Materials for Independent Work of Higher Education Students of the Second (Master's) Level of Higher Education, Specialty 122 Computer Science.* Zhytomyr: Zhytomyr Ivan Franko State University Publishing House, 2024, 60 p.

3. Pavlenko P. M., Filonenko S. F., Cherednikov O. M., and Treitiak V. V., *Mathematical Modeling of Systems and Processes: Textbook.* Kyiv: NAU, 2017, 392 p.

МІЖНАРОДНА НАУКОВА КОНФЕРЕНЦІЯ "СУЧАСНІ ПРОБЛЕМИ МАТЕМАТИЧНОГО АНАЛІЗУ І ТЕОРІЇ ФУНКЦІЙ"

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