



On unclosed mappings with integral constraints

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Abstract. This paper investigates the class of mappings that satisfy the upper weighted Poletsky inequality. We focus on the setting where the mapping is not assumed to be boundary-preserving, and the corresponding majorant satisfies the integral-type restrictions. By imposing specific geometric constraints on the definition domain and respective cluster sets of mappings, we establish the equicontinuity of the corresponding families of mappings in the closure of a domain.

Анотація. Досліджується клас відображень, що задовольняють верхню вагову нерівність Полецького. Ми досліджуємо випадок, коли відображення не зберігає межу області визначення, а відповідна мажоранта задовольняє обмеження інтегрального типу. За певних геометричних обмежень на область визначення та відповідні множини точок накопичення відображень, нами встановлена одностайна неперервність відповідних сімей відображень у замиканні області.

1. INTRODUCTION

The current study falls within the framework of the theory of mappings with bounded and finite distortion (cf. [2, 10, 12, 13, 17, 23, 26, 34, 35]). More precisely, our interest centers on classes of mappings whose very definition incorporates upper bounds for the distortion of the modulus of path families (see, e.g., [1, 3, 13, 30]). Generally, traditional approaches to analyzing the local behavior of such mappings require imposing either homeomorphicity or openness, discreteness, and closedness. Works such as [1, 16, 20, 21, 34] are representative of the former direction, whereas the latter property is investigated in [3, 33, 35].

Let X and Y be metric spaces. A mapping $f: X \rightarrow Y$ is called *closed* if $f(A)$ is closed in $f(X)$ whenever A is closed in X . Otherwise, we mean that f is *unclosed*. Crucially, the combination of openness, discreteness,

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and closedness guarantees that the boundary is preserved under the mapping; conversely, any open, discrete, and boundary-preserving mapping is necessarily closed, as shown in [35, Theorem]. On the other hand, unclosed (boundary-non-preserving) mappings have not yet been sufficiently studied. This lack of consideration is primarily caused by a highly intricate methodology of analysis, given that the properties of the so-called path liftings under unclosed mappings do not readily align with the behavior of the mappings themselves.

Nevertheless, examples of unclosed mappings are quite widespread. For instance, the analytic function $f(z) = z^4$ on the complex plane defines a closed mapping within the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

but fails to retain this property when restricted to the disk

$$B(1, 1) = \{z \in \mathbb{C} : |z - 1| < 1\}.$$

Furthermore, by virtue of the classical Casorati–Weierstrass or Picard theorems, an isolated singularity of an analytic function is essential if and only if the corresponding mapping is not closed. This stems from the fact that closed mappings preserve boundaries, whereas the cluster set of an analytic function at an essential singularity expands to the entire extended complex plane.

Recently, a new approach was initiated to analyze boundary-non-preserving mappings by localized reductions to subdomains where the mappings become closed. As demonstrated in [5], this reduction can be achieved by prescribing specific constraints on the respective cluster set. In the present paper, we extend this line of research. Following the framework established in [5], our primary objective is to examine the boundary behavior of these mappings, alongside the global behavior of their families up to the closure of the domain.

A foundational result in this direction was established by Näkki and Palka (see [16]).

Theorem 1.1 (Näkki–Palka). *Let $\mathfrak{F}$ be a family of K -quasiconformal mappings of a domain $D \neq \overline{\mathbb{R}^n}$ onto a domain D' and let either D or D' be quasiconformally collared on the boundary. Then $\mathfrak{F}$ is uniformly equicontinuous if and only if each $f \in \mathfrak{F}$ can be extended to a continuous mapping of $\overline{D}$ onto $\overline{D'}$ and $\inf_{\mathfrak{F}} h(f(A)) > 0$ for some continuum A in D .*

Here $h(f(A))$ denotes the chordal diameter of the set $f(A) \subset \overline{\mathbb{R}^n}$, see e.g. [34, Definition 12.1].

We have obtained some analogs of this statement, including for homeomorphisms and open discrete and closed mappings, see, e.g. [28] and [32].

In the most general case, when the mappings are neither homeomorphisms nor closed mappings, we have obtained the following statement, see below. We now give some definitions.

A Borel function $\rho: \mathbb{R}^n \rightarrow [0, \infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation $\int_{\gamma} \rho(x) |dx| \geq 1$ holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \text{adm}\Gamma$. Let $\alpha \geq 1$, then α -modulus of Γ is defined by the equality

$$M_{\alpha}(\Gamma) = \inf_{\rho \in \text{adm}\Gamma} \int_{\mathbb{R}^n} \rho^{\alpha}(x) \, dm(x).$$

We set $M(\Gamma) := M_n(\Gamma)$. Let $x_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$,

$$\begin{aligned} S(x_0, r) &= \{x \in \mathbb{R}^n : |x - x_0| = r\}, \\ B(x_0, r) &= \{x \in \mathbb{R}^n : |x - x_0| < r\} \end{aligned} \quad (1.1)$$

and

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}.$$

Given sets $E, F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ a family of all paths $\gamma: [a, b] \rightarrow \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$.

Let $S_i = S(x_0, r_i)$, $i = 1, 2$, where spheres $S(x_0, r_i)$ centered at x_0 of the radius r_i are defined in (1.1). Let $Q: \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. Let $\alpha \geq 1$. A mapping $f: D \rightarrow \overline{\mathbb{R}^n}$ is called a *ring Q -mapping at the point $x_0 \in \overline{D} \setminus \{\infty\}$ with respect to α -modulus*, if the condition

$$M_{\alpha}(f(\Gamma(S_1, S_2, D))) \leq \int_{A \cap D} Q(x) \cdot \eta^{\alpha}(|x - x_0|) \, dm(x) \quad (1.2)$$

holds for some $R_0(x_0) > 0$, all $0 < r_1 < r_2 < R_0$ and all Lebesgue measurable functions $\eta: (r_1, r_2) \rightarrow [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \quad (1.3)$$

The relation (1.2) may be defined at $x_0 = \infty$ with the help of the inversion: by the definition, we say that f satisfies (1.2) whenever $\tilde{f} = f \circ \psi$ satisfies (1.2) at $x_0 = 0$, while $\psi(x) = \frac{x}{|x|^2}$.

A domain $D \subset \mathbb{R}^n$, $n \geq 2$, is called a *uniform domain* with respect to α -modulus, $\alpha \geq 1$, if, $\forall r > 0$, there is $\delta > 0$ with $M_{\alpha}(\Gamma(F, F^*, D)) \geq \delta$ whenever F and F^* are continua of D with $h(F) \geq r$ and $h(F^*) \geq r$. Let I be some set of indices. Domains D_i , $i \in I$, are said to be *equi-uniform*

domains with respect to α -modulus, if, for $r > 0$, the modulus condition above is satisfied by each D_i with the same number δ . It should be noted that the proposed concept of a uniform domain has, generally speaking, no relation to the definition introduced for the uniform domain in the Martio–Sarvas sense [14].

Following [19], a *condenser* in $\mathbb{R}^n$ is a pair $E = (A, C)$, where A is open in $\mathbb{R}^n$ and $C \neq \emptyset$ is a compact subset of A . If $1 \leq \alpha < \infty$, the α -*capacity* of E is defined by

$$\text{cap}_\alpha E = \inf \int_A |\nabla u|^\alpha \, dm,$$

where the infimum is taken over all nonnegative functions u in $\text{ACL}^\alpha(A)$ with compact support in A and $u|_C \geq 1$. Let F be a compact set in $\mathbb{R}^n$. We say that F is of α -*capacity zero* if $\text{cap}_\alpha(A, F) = 0$ for some (and hence for all) bounded open set $A \supset F$. An arbitrary set $E \subset \mathbb{R}^n$ is of α -*capacity zero* if the same is true for every compact subset of E . In this case we write $\text{cap}_\alpha E = 0$ ($\text{cap} E = 0$ if $\alpha = n$), otherwise $\text{cap}_\alpha E > 0$.

Let $Q: \mathbb{R}^n \rightarrow [0, \infty]$ be a Lebesgue measurable function. We set

$$Q'(x) = \begin{cases} Q(x), & Q(x) \geq 1, \\ 1, & Q(x) < 1. \end{cases}$$

Denote by q'_{x_0} the mean value of $Q'(x)$ over the sphere $|x - x_0| = r$, that is,

$$q'_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x-x_0|=r} Q'(x) \, d\mathcal{H}^{n-1},$$

where ω_{n-1} denotes the area of the unit sphere $\mathbb{S}^{n-1} = S(0, 1)$.

Given a domain $D \subset \mathbb{R}^n$, a constant $M > 0$, and an increasing convex function $\Phi: [0, \infty] \rightarrow [0, \infty]$, let us consider the relation

$$\int_D \Phi(Q_f(x)) \cdot \frac{dm(x)}{(1 + |x|^2)^n} \leq M, \quad (1.4)$$

where the meaning of Q_f will be explained below. Let $f: D \rightarrow \mathbb{R}^n$ be a mapping, $E \subset D$ be an arbitrary subset, and $y \in \mathbb{R}^n$. The counting function

$$N(y, f, E) = \text{card}\{x \in E : f(x) = y\}$$

defines the *multiplicity function* of f in E . Accordingly, the global multiplicity of covering over the set E is given by the supremum

$$N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E). \quad (1.5)$$

The following definitions and designations are used further in the article:

Notation	Role / Description
α	order of the modulus of families of paths
p	order of the modulus of families of surfaces
E	a closed set in D containing the pre-image of the cluster set
E_*	a closed set in $\overline{\mathbb{R}^n}$ containing the cluster set $C(f, \partial D)$
F	some closed set, omitted by some family of mappings, having positive capacity for $\alpha = n$, and arbitrary while $\alpha \neq n$
$K_{I,\alpha}(x, f)$	inner dilatation of f of the order α
$Q: D \rightarrow [0, \infty]$	Lebesgue measurable function
Φ	some nonnegative function controlling functions $Q_f(x)$ in (1.4) or $K_{I,\alpha}(x, f)$ in dependence of context
$M > 0$	the constant from (1.4)
δ	a number appearing in the normalized conditions $h(K_f) \geq \delta$ and $h(f^{-1}(K_f), \partial D) \geq \delta > 0$
i_1	$C(f, \partial D) \subset E_*$,
i_2	for each component K of $D'_f \setminus E_*$, where $D'_f := f(D)$, there exists a continuum $K_f \subset K$ satisfying $h(K_f) \geq \delta$ and $h(f^{-1}(K_f), \partial D) \geq \delta > 0$
i_3	$f^{-1}(E_*) \subset E$
$\mathfrak{R}_{Q,\delta,\alpha}^{E_*,E,F}(D)$ or $\mathfrak{R}$	the family of open discrete mappings $f: D \rightarrow \overline{\mathbb{R}^n} \setminus F$ satisfying the condition (1.2)–(1.3) at any $x_0 \in \overline{D}$ such that i_1 – i_3 hold
$\mathfrak{B}_{M,\Phi,\delta,\alpha}^{E_*,E,F}(D)$ or $\mathfrak{B}$	the family of open discrete mappings $f: D \rightarrow \overline{\mathbb{R}^n} \setminus F$ for which there exists a Lebesgue measurable function $Q = Q_f: D \rightarrow [0, \infty]$ satisfying conditions (1.2)–(1.3) at each $x_0 \in \overline{D}$ for some $R_0 = R_0(x_0) > 0$, such that the relation (1.4) and conditions i_1 – i_3 hold
$N(f, D)$	maximal multiplicity of the mapping f in D
N	some number for which $N(f, D) \leq N$
φ	the function from the definition of the Orlicz-Sobolev class $W_{\text{loc}}^{1,\varphi}$
$\mathfrak{DS}_{\varphi,\Phi,\delta,\alpha}^{E_*,E,F,N,M}(D)$ or $\mathfrak{DS}$	the family of bounded open discrete mappings $f: D \rightarrow \overline{\mathbb{R}^n} \setminus F$ in $W_{\text{loc}}^{1,\varphi}(D)$ for which there exists a Lebesgue measurable function $Q = Q_f: D \rightarrow [0, \infty]$ such that $K_{I,\alpha}(x, f) \leq Q(x)$, that is, (1.4) holds, $N(f, D) \leq N$, and conditions i_1 – i_3 are satisfied

To streamline the formulations of our main results, we specify the following geometric conditions on the domain D and the underlying set E :

- (C₁) The set E is nowhere dense in D , and D is finitely connected on E .
- (C₂) For each boundary point $x_0 \in \partial D$, there exists a finite integer $m = m(x_0) \geq 1$ such that every neighborhood U of x_0 contains a subneighborhood $V \subset U$ for which $V \cap D$ is connected and $(V \cap D) \setminus E$ consists of at most m components.

The baseline result for fixed majorants can be formulated as follows (see [5, Theorem 6.1], cf. [7, Theorem 4.1]).

Theorem 1.2 (B.). *Let $\alpha \geq 1$, let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let conditions (C₁)–(C₂) be satisfied. Assume that the set F has positive capacity for $\alpha = n$, and is an arbitrary closed set for $n - 1 < \alpha < n$.*

Suppose also that, at each $x_0 \in \partial D$, the function Q satisfies at least one of the conditions:

- (C₃⁽¹⁾) Q has a finite mean oscillation at x_0 ;
- (C₃⁽²⁾) $q_{x_0}(r) = O([\log \frac{1}{r}]^{n-1})$ as $r \rightarrow 0$;
- (C₃⁽³⁾) the integral condition

$$\int_0^{\delta(x_0)} \frac{dt}{t^{\frac{n-1}{\alpha-1}} q'_{x_0} \frac{1}{\alpha-1}(t)} = \infty$$

holds for some $\delta(x_0) > 0$.

If the family of components of $D_f' \setminus E_$ is equi-uniform over $f \in \mathfrak{R}$ with respect to α -modulus, then every mapping $f \in \mathfrak{R}$ can be continuously extended to ∂D , and the extended family $\mathfrak{R}$ is equicontinuous in $\overline{D}$.*

Our primary objective is to extend Theorem B to the case of varying majorants subject to integral-type restrictions. Similar functional constraints have been successfully applied to other problems, such as the compactness of solution classes for the Dirichlet problem (see, e.g., [21, 22]).

Our main result is established in the following assertion.

Theorem 1.3. *Let $n \geq 2$, $n - 1 < \alpha \leq n$, and let D be a domain in $\mathbb{R}^n$ satisfying conditions (C₁)–(C₂). Let the set F have positive capacity for $\alpha = n$, and be an arbitrary closed set for $n - 1 < \alpha < n$.*

Suppose that the function Φ satisfies the divergence condition

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{\frac{1}{\alpha-1}}} = \infty \tag{1.6}$$

for some $\delta_0 > \tau_0 := \Phi(0)$. If the family of components of $D'_f \setminus E_*$ is equi-uniform over $f \in \mathfrak{B}$ with respect to α -modulus, then each $f \in \mathfrak{B}$ admits a continuous extension to ∂D , and the corresponding extended family $\overline{\mathfrak{B}}$ is equicontinuous in $\overline{D}$.

To demonstrate the meaningfulness of the above result, in particular, to point out the difference between mappings with a fixed majorant and mappings for which there is no such majorant, consider the following example. Let $x \in B(0, e^{-1})$ and $p = n$, $n \geq 2$. We set

$$Q_m(x) = \begin{cases} m, & |x| \leq e^{-m}, \\ \log \frac{1}{|x|}, & |x| > e^{-m}, \end{cases} \quad m = 2, 3, \dots$$

We now put

$$f_m(x) = \frac{x}{|x|} \exp \left\{ - \int_{|x|}^{e^{-1}} \frac{dt}{t q_{0,m}^{\frac{1}{n-1}}(t)} \right\},$$

where

$$q_{0,m}(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S(0,t)} Q_m(x) \, d\mathcal{H}^{n-1}.$$

Arguing similarly to [13, Proposition 6.3], we observe that f_m satisfy (1.2) and (1.3) with $Q = Q_m$. At the same time, f_m have no fixed majorant Q in the context of these relations.

The conditions $(\mathbf{C}_1)$ – $(\mathbf{C}_2)$ hold because $D = B(0, e^{-1})$ is locally connected on the boundary and $E = \emptyset$. As F we may put any continuum outside the open unit ball $\mathbb{B}^n = B(0, 1)$ because, obviously, $f_m(D) = \mathbb{B}^n$. Finally,

$$E_* = C(f_m, \partial B(0, e^{-1})) = \mathbb{S}^{n-1}.$$

Note that $\mathbb{B}^n$ is a uniform domain, because $\mathbb{B}^n$ is qc-collared (see [34, Theorem 17.12]) and, consequently, is uniform (see [15, Corollary 6.6]). Thus, the family $f_m(D) \setminus E_* = f_m(D) = \mathbb{B}^n$ ($m = 2, 3, \dots$) is equi-uniform. Observe that for the function $\Phi(t) = e^t$ by the Fubini theorem we obtain that

$$\begin{aligned} \int_{B(0, e^{-1})} \Phi(Q_m(x)) \cdot \frac{dm(x)}{(1 + |x|^2)^n} &\leq \int_{B(0, e^{-m})} e^m dm(x) + \int_{B(0, e^{-1})} \exp \log \frac{1}{|x|} dm(x) \\ &= \Omega_n e^{m(1-n)} + \omega_{n-1} \int_0^{e^{-1}} t^{n-1} \cdot \frac{1}{t} dt \\ &\leq \Omega_n + \omega_{n-1} e^{-1} := M, \end{aligned}$$

where Ω_n denotes the volume of the unit ball $\mathbb{B}^n = B(0, 1)$. Obviously, the relation (1.6) holds for $\Phi(t) = e^t$. Thus, we may apply Theorem 1.3 for the family f_m . However, we cannot apply Theorem B, in this case.

2. PRELIMINARIES

The following assertion was established in [5, Lemma 2.1].

Lemma 2.1. *Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $x_0 \in \partial D$. Suppose that E is a closed, nowhere dense subset of D , and assume that D is finitely connected on E , meaning that for each $z_0 \in E$ and every neighborhood $\tilde{U}$ of z_0 , there exists a subneighborhood $\tilde{V} \subset \tilde{U}$ such that $(D \cap \tilde{V}) \setminus E$ breaks into a finite number of components.*

Furthermore, assume the following geometric condition holds: every neighborhood U of x_0 contains a subneighborhood $V \subset U$ such that:

- (a) *the intersection $V \cap D$ is connected,*
- (b) *the set $(V \cap D) \setminus E$ is composed of at most m components, $1 \leq m < \infty$.*

Let $x_k \xrightarrow{k \rightarrow \infty} x_0$ and $y_k \xrightarrow{k \rightarrow \infty} x_0$ with $x_k, y_k \in D \setminus E$ and $k \in \mathbb{N}$. Then one can extract subsequences x_{k_l} and y_{k_l} , contained in a certain sequence of neighborhoods V_l of x_0 such that $V_l \subset B(x_0, 2^{-l})$, and, moreover, each pair x_{k_l} and y_{k_l} can be linked by a path γ_l in $V_l \cap D$, where γ_l intersects E at no more than $m - 1$ points.

Consider a discrete open mapping $f: D \rightarrow \mathbb{R}^n$ defined on a domain $D \subset \mathbb{R}^n$, a path $\beta: [a, b) \rightarrow \mathbb{R}^n$, and a point $x \in f^{-1}(\beta(a))$. A path $\alpha: [a, c) \rightarrow D$ is called a *maximal f -lifting* of β starting at x if:

- (1) $\alpha(a) = x$;
- (2) $f \circ \alpha = \beta|_{[a, c)}$;
- (3) given any $c < c' \leq b$, there is no path $\alpha': [a, c') \rightarrow D$ satisfying $\alpha = \alpha'|_{[a, c)}$ and $f \circ \alpha' = \beta|_{[a, c')}$.

Throughout this paper, a path $\beta: [a, b) \rightarrow \mathbb{R}^n$ is said to converge to a set $C \subset \mathbb{R}^n$ as $t \rightarrow b$ if $h(\beta(t), C) = \sup_{x \in C} h(\beta(t), x) \xrightarrow{t \rightarrow b} 0$.

Proposition 2.2 ([11, Lemma 3.12]). *Let $f: D \rightarrow \mathbb{R}^n$, $n \geq 2$, be an open discrete mapping, let $x_0 \in D$, and let $\beta: [a, b) \rightarrow \mathbb{R}^n$ be a path starting at $\beta(a) = f(x_0)$. Assume also, that either $\lim_{t \rightarrow b} \beta(t)$ exists or $\beta(t) \xrightarrow{t \rightarrow b} \partial f(D)$.*

Then β possesses a maximal f -lifting $\alpha: [a, c) \rightarrow D$ with $\alpha(a) = x_0$. If $\alpha(t) \xrightarrow{t \rightarrow c} x_1 \in D$, then $c = b$ and $f(x_1) = \lim_{t \rightarrow b} \beta(t)$. In the alternative case, $\alpha(t) \xrightarrow{t \rightarrow c} \partial D$.

Lemma 2.3 ([29, Lemma 2.1]). *Let $1 \leq \alpha \leq n$ and $\Phi: [0, \infty] \rightarrow [0, \infty]$ be a strictly increasing convex function such that the relation (1.6) holds for some $\delta_0 > \tau_0 := \Phi(0)$. Let $\mathfrak{Q}$ be a family of functions $Q: \mathbb{R}^n \rightarrow [0, \infty]$ such that*

$$\int_D \Phi(Q(x)) \frac{dm(x)}{(1 + |x|^2)^n} \leq M_0 < \infty \quad (2.1)$$

for some $0 < M_0 < \infty$. Now, for any $0 < R_0 < 1$ and for every $\sigma > 0$ there exists $0 < r_ = r_*(\sigma, R_0, \Phi) < R_0$ such that*

$$\int_{\varepsilon}^{R_0} \frac{dt}{t^{\frac{n-1}{\alpha-1}} q_{x_0}^{\frac{1}{\alpha-1}}(t)} \geq \sigma, \quad \varepsilon \in (0, r_*),$$

for any $Q \in \mathfrak{Q}$.

Remark 2.4. We set $Q'(x) = \begin{cases} Q(x), & Q(x) \geq 1 \\ 1, & Q(x) < 1. \end{cases}$

Notice that $Q'(x)$ satisfies (2.1) up to a constant, see [29, Theorem 1.3].

3. PROOF OF THEOREM 1.3

The core of our verification is based on the topological and modulus techniques developed in [5]. First, the equicontinuity of the class $\mathfrak{B}$ inside the domain D follows directly from [29, Theorems 1.1, 1.2]. Thus, our primary task is to establish the uniform boundary behavior. Specifically, we shall prove that for each finite boundary point $x_0 \in \partial D$ and any given number $\varepsilon > 0$, there exists a positive radius $\delta = \delta(\varepsilon, x_0) > 0$ ensuring that the inequality

$$h(f(x), f(y)) < \varepsilon \quad (3.1)$$

holds for all mappings $f \in \mathfrak{B}$ and all points $x, y \in B(x_0, \delta) \cap D$.

Supposing the contrary, we assume that condition (3.1) fails. Then one can find a boundary point $x_0 \in \partial D$, a positive constant $\mu_0 > 0$, a sequence of mappings $f_k \in \mathfrak{B}$, and pairs of points $x_k, y_k \in D$ converging to x_0 as $k \rightarrow \infty$, such that

$$h(f_k(x_k), f_k(y_k)) \geq \frac{\mu_0}{2}. \quad (3.2)$$

By virtue of assumption i_2 , there exists a system of shrinking neighborhoods $V_k \subset B(x_0, 2^{-k})$ ($k \in \mathbb{N}$) where each intersection $V_k \cap D$ is connected, and the complement $(V_k \cap D) \setminus E$ consists of at most m components, $1 \leq m < \infty$. Applying Cantor's diagonal argument, we can assume that the points x_k and y_k are chosen outside the exceptional set E . Under these conditions, Lemma 2.1 guarantees the existence of subsequences x_{k_j} and y_{k_j} located within a dedicated sequence of neighborhoods V_j around

x_0 with $\text{diam } V_j \xrightarrow{j \rightarrow \infty} 0$. Moreover, each pair x_{k_j}, y_{k_j} may be joined by a path γ_j inside $V_j \cap D$ that contains at most $m - 1$ points of E . Standard subsequence reduction allows us to retain the original indices k for these properties, yielding joining paths $\gamma_k: [0, 1] \rightarrow D$ with $\gamma_k(0) = x_k$ and $\gamma_k(1) = y_k$ for $k \in \mathbb{N}$.

Let us observe that the image path $f_k(\gamma_k)$ contains no more than $m - 1$ points of the target set E_* . Indeed, the presence of m or more such points, say $b_1, \dots, b_m$, would imply that their preimages lie simultaneously on γ_k and inside the set $f_k^{-1}(E_*) \subset E$, which directly violates the fact that γ_k meets E at most at $m - 1$ points.

We represent the intersection points $f_k(\gamma_k) \cap E_*$ as:

$$b_1 = f_k(\gamma_k(t_1)), \quad b_2 = f_k(\gamma_k(t_2)), \quad \dots \quad b_j = f_k(\gamma_k(t_j)),$$

ordered by the parameter values

$$t_0 := 0 \leq t_1 \leq t_2 \leq \dots \leq t_j \leq 1 = t_{j+1},$$

where $0 \leq j \leq m - 1$. Applying the triangle inequality to the initial metric estimate (3.2), we obtain

$$\frac{\mu_0}{2} \leq h(f_k(x_k), f_k(y_k)) \leq \sum_{r=0}^j h(f_k(\gamma_k(t_r)), f_k(\gamma_k(t_{r+1}))). \quad (3.3)$$

The sum in (3.3) implies that for each index k , there is a positive integer $r(k) \leq m - 1$ for which

$$h(f_k(\gamma_k(t_{r(k)})), f_k(\gamma_k(t_{r(k)+1}))) \geq \frac{\mu_0}{2(j+1)} \geq \frac{\mu_0}{2m}. \quad (3.4)$$

The open curve segment $G_k := |f_k(\gamma_k)|_{(t_{r(k)}, t_{r(k)+1})}$ is clearly contained in the complementary set $D'_{f_k} \setminus E_*$. Utilizing the endpoint limits

$$\begin{aligned} a_{s,k} &:= f_k(\gamma_k(p_s)) \xrightarrow{s \rightarrow \infty} w_k := f_k(\gamma_k(t_{r(k)})) \\ b_{s,k} &:= f_k(\gamma_k(q_s)) \xrightarrow{s \rightarrow \infty} f_k(\gamma_k(t_{r(k)+1})) \end{aligned}$$

within the interval $t_{r(k)} < p_s < q_s < t_{r(k)+1}$, the triangle inequality combined with (3.4) yields:

$$\begin{aligned} h(a_{s,k}, b_{s,k}) &\geq h(b_{s,k}, w_k) - h(w_k, a_{s,k}) \\ &\geq h(f_k(\gamma_k(t_{r(k)+1})), w_k) - h(f_k(\gamma_k(t_{r(k)+1})), b_{s,k}) - h(w_k, a_{s,k}) \\ &\geq \frac{\mu_0}{2m} - h(f_k(\gamma_k(t_{r(k)+1})), b_{s,k}) - h(w_k, a_{s,k}). \end{aligned}$$

Letting $s \rightarrow \infty$, the last two terms vanish, which guarantees the existence of a specialized index $s(k) \in \mathbb{N}$ such that the chordal distance satisfies

$$h(a_{s(k),k}, b_{s(k),k}) \geq \frac{\mu_0}{4m}.$$

Thus, the restricted path segment

$$\tilde{G}_k := f_k(\gamma_k)|_{[p_{s(k)}, q_{s(k)}]}$$

defines a continuum in a certain component K_k of the open set $D'_{f_k} \setminus E_*$. Recalling that $h(K_{f_k}) \geq \delta$ by the definition of the class $\mathfrak{B}$, we can exploit the equi-uniformity property of the pair of continua $\tilde{G}_k$ and K_{f_k} inside K_k . For the uniform positive number $\delta_* := \min\{\delta, \frac{\varepsilon}{4m}\}$, there exists a constant $C_0 > 0$ such that

$$M_\alpha(\Gamma(\tilde{G}_k, K_{f_k}, K_k)) \geq C_0 > 0, \quad k \in \mathbb{N}. \quad (3.5)$$

Our next step is to demonstrate that the uniform lower bound (3.5) contradicts the integral convergence conditions. Consider the path family Γ_k of paths $\beta_k: [a, b] \rightarrow \overline{\mathbb{R}^n}$ such that $\beta_k(a) \in \tilde{G}_k$, $\beta_k(t) \in K_k$ for all $t \in [a, b]$, and $\lim_{t \rightarrow b-0} \beta_k(t) := B_k \in K_{f_k}$. Relation (3.5) implies

$$M_\alpha(\Gamma_k) = M_\alpha(\Gamma(\tilde{G}_k, K_{f_k}, K_k)) \geq C_0 > 0, \quad k \in \mathbb{N}. \quad (3.6)$$

Let Γ'_k denote the family of maximal f_k -liftings $\alpha_k: [a, c] \rightarrow D$ of the family Γ_k starting at $|\gamma_k|$, whose existence is guaranteed by Proposition 2.2.

Let us rule out the possibility that these liftings approach the boundary, i.e., $\alpha_k(t) \xrightarrow{t \rightarrow c-0} \partial D$. Assuming the contrary and choosing a sequence $\varphi_m \in [0, c)$ converging to $c - 0$, the compactness of ∂D yields a sequence of points $w_m \in \partial D$ such that

$$h(\alpha_k(\varphi_m), \partial D) = h(\alpha_k(\varphi_m), w_m) \xrightarrow{m \rightarrow \infty} 0.$$

Extracting a cluster point $w_m \rightarrow w_0 \in \partial D$, the triangle inequality yields

$$h(\alpha_k(\varphi_m), w_0) \leq h(\alpha_k(\varphi_m), w_m) + h(w_m, w_0) \xrightarrow{m \rightarrow \infty} 0. \quad (3.7)$$

Simultaneously, the continuity of the mapping f_k implies that

$$f_k(\alpha_k(\varphi_m)) = \beta_k(\varphi_m) \xrightarrow{m \rightarrow \infty} \beta(c), \quad (3.8)$$

because the paths $\beta_k(t)$ are restricted to $K_k \subset D'_{f_k} \setminus E_*$. Combining (3.7) and (3.8), we see that the cluster point satisfies $\beta_k(c) \in C(f_k, \partial D) \subset E_*$. This directly contradicts the fact that the path β_k is contained in $D' \setminus E_*$, thus ruling out the boundary-bound scenario for the liftings.

Therefore, Proposition 2.2 implies that the liftings $\alpha_k: [a, b] \rightarrow D$ are complete, terminating at points $x_1 \in D$ such that $f_k(\alpha_k(b)) = f_k(x_1)$, which

means $\alpha_k(b) \in f_k^{-1}(K_{f_k})$. By the geometric definition of the mapping class, we have

$$h(f_k^{-1}(K_{f_k}), \partial D) \geq \delta > 0. \quad (3.9)$$

Since x_0 is finite, condition (3.9) ensures the inclusion

$$f_k^{-1}(K_{f_k}) \subset D \setminus B(x_0, R_0) \quad (3.10)$$

for a fixed radius $R_0 > 0$. Choosing k large enough so that $2^{-k} < \mu_0$ and ensuring $\mu_0 < R_0$, property (3.10) implies the following path minorization in the sense of Kuratowski (see [9, §46, Theorem 1.I.5]):

$$\Gamma'_k > \Gamma(S(x_0, 2^{-k}), S(x_0, \mu_0), D). \quad (3.11)$$

Let us analyze the spherical average integral

$$I_k = I(x_0, 2^{-k}, \mu_0) = \int_{2^{-k}}^{\mu_0} \frac{dr}{r^{\frac{n-1}{\alpha-1}} \tilde{q}_{k,x_0}^{\frac{1}{\alpha-1}}(r)}, \quad (3.12)$$

where

$$\tilde{q}_{k,x_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(x_0, r)} \tilde{Q}_{f_k}(x) d\mathcal{H}^{n-1}$$

and $\tilde{Q}_{f_k}(x) = \max\{Q_{f_k}(x), 1\}$.

By Lemma 2.3 and Remark 2.4, we have the asymptotic divergence

$$I_k = \int_{2^{-k}}^{\mu_0} \frac{dr}{r^{\frac{n-1}{\alpha-1}} \tilde{q}_{k,x_0}^{\frac{1}{\alpha-1}}(r)} \xrightarrow{k \rightarrow \infty} \infty. \quad (3.13)$$

Since $\tilde{q}_{k,x_0}(r) \geq 1$, the integral I_k is positive and finite for all sufficiently large k .

We put

$$\psi_k(t) = \begin{cases} \left(t^{\frac{n-1}{\alpha-1}} \tilde{q}_{k,x_0}^{\frac{1}{\alpha-1}}(t) \right)^{-1}, & t \in (2^{-k}, \mu_0), \\ 0, & t \notin (2^{-k}, \mu_0). \end{cases}$$

Defining the normalized weight as $\eta_k(t) = \frac{\psi_k(t)}{I_k}$, we obtain an admissible function for relation (1.3) on the interval $(2^{-k}, \mu_0)$.

By applying condition (1.2), the structural path relation (3.11), the divergence property (3.13) and Fubini's theorem, we obtain the upper bound

$$\begin{aligned} M_\alpha(f_k(\Gamma'_k)) &\leq \frac{1}{I_k^\alpha} \int_{A(x_0, 2^{-k}, \mu_0)} \tilde{Q}_{f_k}(x) \cdot \psi_k^\alpha(|x - x_0|) dm(x) \\ &= \frac{\omega_{n-1}}{I_k^{\alpha-1}} \xrightarrow{k \rightarrow \infty} 0 \end{aligned} \quad (3.14)$$

Observe that $\Gamma_k > f_k(\Gamma'_k)$ and, therefore, $M_\alpha(\Gamma_k) \leq M_\alpha(f_k(\Gamma'_k))$ due to the minorization of the modulus. Consequently, combining (3.14) with relations (3.11)–(3.12) yields the asymptotic estimate

$$M_\alpha(\Gamma_k) \leq M_\alpha(f_k(\Gamma'_k)) \leq \Delta(k) \xrightarrow{k \rightarrow \infty} 0. \quad (3.15)$$

The asymptotic decay in (3.15) directly contradicts the uniform strictly positive lower bound established in (3.6), which completes the verification of the boundary property (3.1).

Note that the uniform estimate (3.1) immediately guarantees that every mapping $f \in \mathfrak{B}$ can be continuously extended to the finite boundary point x_0 . Indeed, if the extension were impossible, one could find sequences $x_k, y_k \rightarrow x_0$ such that $h(f(x_k), f(y_k)) \geq \mu_0$, which is a direct violation of (3.1).

To establish that family $\overline{\mathfrak{B}}$ of the extended mappings is equicontinuous at x_0 with respect to the closure $\overline{D}$, we again suppose the contrary. This implies the existence of a point $x_0 \in \partial D$, a constant $\mu_0 > 0$, a sequence $x_k \in \overline{D}$ approaching x_0 , and mappings $f_k \in \overline{\mathfrak{B}}$ such that

$$h(f_k(x_k), f_k(x_0)) \geq \mu_0. \quad (3.16)$$

By the continuity of the boundary extensions, we may assume that $x_k \in D$. Furthermore, relation (3.16) allows us to find an auxiliary sequence $x'_k \in D$ that converges to x_0 and satisfies

$$h(f_k(x_k), f_k(x'_k)) \geq \frac{\mu_0}{2}.$$

This inequality directly contradicts the uniform boundary estimate (3.1), which fully settles the case for finite points $x_0 \neq \infty$.

Finally, consider the case where $x_0 = \infty$. We introduce the family $\tilde{f}_k := \psi \circ f_k$, where $\psi(y) = \frac{y}{|y|^2}$ denotes the standard inversion. By the hypotheses of Theorem 1.3, these transformed mappings satisfy the conditions (1.2) and (1.3) at the origin under the modified majorant

$$\tilde{Q}(x) := Q_{f_k}\left(\frac{x}{|x|^2}\right).$$

The conformal change of variables $x = \frac{y}{|y|^2}$ generates the standard Jacobian transformation $J(x, y) = \frac{1}{|y|^{2n}}$, yielding the valid integral transformation:

$$\int_{\mathbb{R}^n} \Phi\left(Q_{f_k}\left(\frac{x}{|x|^2}\right)\right) \cdot \frac{dm(x)}{(1 + |x|^2)^n} = \int_{\mathbb{R}^n} \Phi(Q_{f_k}(z)) \cdot \frac{dm(z)}{(1 + |z|^2)^n} < \infty. \quad (3.17)$$

The integral identity (3.17) guarantees that the variational modulus technique may be applied to the majorant $Q_{f_k}(\frac{x}{|x|^2})$ exactly as done for $Q_{f_k}(x)$. Replicating the entire analytical procedure starting from equation (3.2) for

the inverted family $\tilde{f}_k$, we arrive at an identical contradiction, which completes the proof of Theorem 1.3. $\square$

4. SOBOLEV AND ORLICZ-SOBOLEV CLASSES

Let ω be an open set in the extended Euclidean space $\overline{\mathbb{R}^k} := \mathbb{R}^k \cup \{\infty\}$ for some $k \in \{1, \dots, n-1\}$. A continuous mapping $S: \omega \rightarrow \mathbb{R}^n$ defines a k -dimensional surface S in $\mathbb{R}^n$, and its image $S(\omega) \subseteq \mathbb{R}^n$ is frequently referred to as the surface S itself. The counting function

$$N(S, y) = \text{card} S^{-1}(y) = \text{card}\{x \in \omega : S(x) = y\}$$

stands for the *multiplicity function* of S at a point $y \in \mathbb{R}^n$, representing the covering order of y by the surface S . It is a well-established property that this multiplicity function exhibits lower semi-continuity, meaning that

$$N(S, y) \geq \liminf_{m \rightarrow \infty} N(S, y_m)$$

holds for any sequence $y_m \in \mathbb{R}^n$, $m = 1, 2, \dots$, converging to $y \in \mathbb{R}^n$ as $m \rightarrow \infty$ (see, e.g., [18, p. 160]). Consequently, $N(S, y)$ belongs to the class of Borel measurable functions and is therefore measurable with respect to any Hausdorff measure $\mathcal{H}^k$ (cf. [24, p. 52]).

For any Borel function $\rho: \mathbb{R}^n \rightarrow [0, \infty]$, its *integral over the k -dimensional surface S* in $\mathbb{R}^n$ ($n \geq 2$) is introduced via the identity

$$\int_S \rho \, d\mathcal{A} := \int_{\mathbb{R}^n} \rho(y) N(S, y) \, d\mathcal{H}^k y.$$

Let $\mathcal{S}$ denote a collection of such k -dimensional surfaces in $\mathbb{R}^n$. A Borel function $\rho: \mathbb{R}^n \rightarrow [0, \infty]$ is called *admissible* for $\mathcal{S}$ (written as $\rho \in \text{adm}\mathcal{S}$) provided that

$$\int_S \rho^k \, d\mathcal{A} \geq 1$$

is satisfied for each $S \in \mathcal{S}$. Given an exponent $p \in [k, \infty)$, the *p -modulus* of the family $\mathcal{S}$ is defined as the quantity

$$M_p(\mathcal{S}) = \inf_{\rho \in \text{adm}\mathcal{S}} \int_{\mathbb{R}^n} \rho^p(x) \, dm(x).$$

We now introduce a class of mappings that generalizes quasiconformal mappings in terms of the Gehring ring definition (see [6], cf. [13, Chapter 9]). Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $x_0 \in \overline{D} \setminus \{\infty\}$. Given a Lebesgue measurable function $Q: D \rightarrow (0, \infty)$, a mapping $f: D \rightarrow D'$ is called a

lower Q -mapping at x_0 relative to p -modulus if the modulus inequality

$$M_p(f(\Sigma_\varepsilon)) \geq \inf_{\rho \in \text{ext}_p \text{ adm } \Sigma_\varepsilon} \int_{D \cap A(x_0, \varepsilon, R_0)} \frac{\rho^p(x)}{Q(x)} dm(x) \quad (4.1)$$

holds for every spherical ring

$$A(x_0, \varepsilon, R_0) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < R_0 \in (0, d_0)\},$$

$$d_0 = \sup_{x \in D} |x - x_0|,$$

where Σ_ε is the family of all intersections of the spheres $S(x_0, r)$ with the domain D , $r \in (\varepsilon, R_0)$. If $p = n$, we say that f is a lower Q -mapping at x_0 . We say that f is a lower Q -mapping relative to p -modulus in $A \subset \mathbb{R}^n$ if (4.1) is true for all $x_0 \in A$.

Following the terminology established in [19, Section II.10], a *condenser* in $\mathbb{R}^n$ is defined as a pair $E = (A, C)$, where $A \subset \mathbb{R}^n$ is an open set and $C \subset A$ is a non-empty compact subset. For any exponent $\alpha \geq 1$, the α -capacity of such a condenser E is given by the variational infimum

$$\text{cap}_\alpha E = \text{cap}_\alpha(A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^\alpha dm(x),$$

where the class of admissible test functions $W_0(E) = W_0(A, C)$ consists of all non-negative, absolutely continuous on lines (ACL) functions $u: A \rightarrow \mathbb{R}^1$ with compact support $u \in C_0(A)$ such that $u(x) \geq 1$ for all $x \in C$. In this context, ∇u denotes the standard gradient vector, while

$$|\nabla u| = \left(\sum_{i=1}^n (\partial_i u)^2 \right)^{\frac{1}{2}}.$$

The functional framework and boundary behavior established below rely significantly on the following auxiliary result (see [30, Lemma 4.4], cf. [31, Lemma 4.2]).

Lemma 4.1. *Let $x_0 \in D$, $n \geq 2$, $p > n - 1$, and let $Q: D \rightarrow [0, \infty]$ be a locally integrable function of degree $s := \frac{n-1}{p-n+1}$. If an open discrete mapping $f: D \rightarrow \overline{\mathbb{R}^n}$ is a lower Q -mapping at x_0 with respect to p -modulus, then the capacity inequality*

$$\text{cap}_\alpha(f(E)) \leq \frac{\omega_{n-1}}{I^{*\alpha-1}}$$

holds at x_0 , where the exponents are given by

$$\alpha := \frac{p}{p-n+1} \quad \text{and} \quad Q^* = Q^{\frac{n-1}{p-n+1}} = Q^s.$$

Here, E denotes the condenser $(B(x_0, r_2), \overline{B(x_0, r_1)})$, $q_{x_0}^*(r)$ stands for the spherical mean of $Q^{\frac{n-1}{p-n+1}}$ over the sphere $|x - x_0| = r$, and the integral I^* is defined as

$$I^* = I^*(r_1, r_2) = \int_{r_1}^{r_2} \frac{dt}{t^{\frac{n-1}{\alpha-1}} q_{x_0}^* \frac{1}{\alpha-1}(t)}$$

for radii satisfying $0 < r_1 < r_2 < \text{dist}(x_0, \partial D)$.

To establish our main convergence results, we will utilize the following variational minimization property (see [13, Lemma 7.4, ch. 7] for the case $\alpha = n$, and [25, Lemma 2] when $\alpha \neq n$).

Proposition 4.2. *Let $x_0 \in \mathbb{R}^n$, and let $Q: \mathbb{R}^n \rightarrow [0, \infty]$ be a Lebesgue measurable function such that $Q \in L_{\text{loc}}^1(\mathbb{R}^n)$. Consider the spherical ring $A := A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$ and introduce the extremal weight function*

$$\eta_0(r) = \frac{1}{I r^{\frac{n-1}{\alpha-1}} q_{x_0}^* \frac{1}{\alpha-1}(r)},$$

where the normalization integral is defined by

$$I = I(x_0, r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{\alpha-1}} q_{x_0}^* \frac{1}{\alpha-1}(r)}$$

and

$$q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x-x_0|=r} Q(x) \, d\mathcal{H}^{n-1}$$

represents the integral average of Q over the sphere $S(x_0, r)$. Then the following minimization property holds:

$$\frac{\omega_{n-1}}{I^{\alpha-1}} = \int_A Q(x) \cdot \eta_0^\alpha(|x - x_0|) \, dm(x) \leq \int_A Q(x) \cdot \eta^\alpha(|x - x_0|) \, dm(x) \quad (4.2)$$

for any Lebesgue measurable function $\eta: (r_1, r_2) \rightarrow [0, \infty]$ satisfying

$$\int_{r_1}^{r_2} \eta(r) \, dr = 1.$$

Moreover, estimate (4.2) remains valid for any such weight functions η obeying the inequality $\int_{r_1}^{r_2} \eta(r) \, dr \geq 1$, (cf. [30, Remark 3.1]).

Combining Lemma 4.1 with Proposition 4.2, we obtain the following statement.

Lemma 4.3. *Let $x_0 \in D$, let $n \geq 2$, let $p > n - 1$ and let $Q: D \rightarrow [0, \infty]$ be a locally integrable function with degree $s := \frac{n-1}{p-n+1}$. If $f: D \rightarrow \overline{\mathbb{R}^n}$ is an open discrete lower Q -mapping at x_0 with respect to p -modulus, then f satisfies*

$$\text{cap}_\alpha(f(E)) \leq \int_{A(x_0, r_1, r_2)} Q^{\frac{n-1}{p-n+1}}(x) \cdot \eta^\alpha(|x - x_0|) \, dm(x)$$

for any Lebesgue measurable function $\eta: (r_1, r_2) \rightarrow [0, \infty]$ with

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1,$$

where $\alpha := \frac{p}{p-n+1}$ and $Q^* = Q^{\frac{n-1}{p-n+1}} = Q^s$. Here $E = (B(x_0, r_2), \overline{B(x_0, r_1)})$, $0 < r_1 < r_2 < \text{dist}(x_0, \partial D)$.

The following definition is from [19, Section 2, Ch. III]. Let F be a compact set in $\mathbb{R}^n$. We say that F is of α -capacity zero if $\text{cap}_\alpha(A, F) = 0$ for some bounded open set $A \supset F$. An arbitrary set $E \subset \mathbb{R}^n$ is of α -capacity zero if the same is true for every compact subset of E . In this case we write $\text{cap}_\alpha E = 0$ ($\text{cap} E = 0$ if $\alpha = n$), otherwise $\text{cap}_\alpha E > 0$. The following statement was proved in [27, Lemma 4] for $\alpha = n$.

Lemma 4.4. *Let $n-1 < \alpha \leq n$, let D be a domain in $\mathbb{R}^n$, $n \geq 2$, let $E \subset \overline{\mathbb{R}^n}$ be a compact set of positive capacity for $\alpha = n$, and an arbitrary set for $\alpha \neq n$, let $\Phi: [0, \infty] \rightarrow [0, \infty]$ be an increasing convex function, let $M > 0$ and let $\mathfrak{O}D_{\Phi, E}^{\alpha, M}(x_0)$ be a family of open discrete mappings $f: D \rightarrow \overline{\mathbb{R}^n} \setminus E$ for which there exists a Lebesgue measurable function $Q = Q_f: D \rightarrow [0, \infty]$ such that*

$$\text{cap}_\alpha(f(\mathcal{E})) \leq \int_{A(x_0, r_1, r_2)} Q(x) \cdot \eta^\alpha(|x - x_0|) \, dm(x)$$

for any Lebesgue measurable function $\eta: (r_1, r_2) \rightarrow [0, \infty]$ with

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1, \quad \mathcal{E} = (B(x_0, r_2), \overline{B(x_0, r_1)}),$$

while

$$\int_D \Phi(Q_f(x)) \cdot \frac{dm(x)}{(1 + |x|^2)^n} \leq M.$$

If for some $\delta_0 > \tau_0 := \Phi(0)$,

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{\alpha-1}}} = \infty,$$

then the family of mappings $\mathfrak{D}D_{\Phi,E}^{\alpha,M}(x_0)$ is equicontinuous at the point x_0 .

Proof of Lemma 4.4 repeats verbatim the proofs of [29, Theorem 1.1 and Theorem 1.2], and is therefore omitted.

For any point $x \in D$ and fixed exponents $p, q \geq 1$, the *inner dilatation* $K_{I,q}(x, f)$ of order q is introduced by the relation

$$K_{I,q}(x, f) = \begin{cases} \frac{|J(x, f)|}{l(f'(x))^q}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise.} \end{cases} \quad (4.3)$$

The next statement establishes the lower Q -mapping property for open discrete mappings within the Orlicz-Sobolev classes. For the case of homeomorphisms and boundary points $x_0 \in \overline{D}$, this property was originally verified in [8, Theorem 2.1] (cf. [17, Lemma 2.3]).

Lemma 4.5. *Let D be a domain in $\mathbb{R}^n$, $n \geq 3$, and let $\varphi: (0, \infty) \rightarrow (0, \infty)$ be a non-decreasing function satisfying the integral convergence condition*

$$\int_1^{\infty} \left(\frac{t}{\varphi(t)} \right)^{\frac{1}{n-2}} dt < \infty.$$

Suppose that $p > n - 1$. Then every open discrete mapping $f: D \rightarrow \mathbb{R}^n$ of finite distortion belonging to the class $W_{\text{loc}}^{1,\varphi}(D)$ with a finite global multiplicity $N(f, D) < \infty$ is a lower Q -mapping relative to p -modulus at each point $x_0 \in \overline{D}$. The corresponding majorant is given by the function

$$Q(x) = N(f, D) \cdot K_{I,\alpha}^{\frac{p-n+1}{n-1}}(x, f),$$

where $\alpha := \frac{p}{p-n+1}$, the inner dilatation $K_{I,\alpha}(x, f)$ is defined according to the rule (4.3), and $N(f, D)$ is the total multiplicity bound introduced in (1.5).

The following auxiliary assertion will be crucial for further analysis (see, e.g., [30, Theorem 5.6]).

Lemma 4.6. *Let $x_0 \in \partial D$, let $f: D \rightarrow \mathbb{R}^n$ be a bounded, open, discrete, and closed lower Q -mapping with respect to p -modulus in a domain $D \subset \mathbb{R}^n$,*

$$Q \in L_{\text{loc}}^{\frac{n-1}{p-n+1}}(\mathbb{R}^n), \quad n-1 < p, \quad \alpha := \frac{p}{p-n+1}.$$

Then, for every

$$\varepsilon_0 < d_0 := \sup_{x \in D} |x - x_0|$$

and every compact set $C_2 \subset D \setminus B(x_0, \varepsilon_0)$ there exists ε_1 , $0 < \varepsilon_1 < \varepsilon_0$, such that, for each $\varepsilon \in (0, \varepsilon_1)$ and each compact $C_1 \subset \overline{B(x_0, \varepsilon)} \cap D$ the inequality

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leq \int_{A(x_0, \varepsilon, \varepsilon_1)} Q^{\frac{n-1}{p-n+1}}(x) \eta^\alpha(|x - x_0|) \, dm(x) \quad (4.4)$$

holds, where

$$A(x_0, \varepsilon, \varepsilon_1) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_1\}$$

and $\eta : (\varepsilon, \varepsilon_1) \rightarrow [0, \infty]$ is an arbitrary Lebesgue measurable function such that

$$\int_{\varepsilon}^{\varepsilon_1} \eta(r) \, dr = 1.$$

To formulate the boundary extension property for this class, we introduce a slight modification of our geometric assumptions:

(**C**₁^{*}) The set E is nowhere dense in D , and D is finitely connected on $E \cup \partial D$.

Theorem 4.7. Let $\alpha \in (n-1, n]$, let D be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let conditions (**C**₁^{*}) and (**C**₂) be satisfied. Assume that the function Φ satisfies the condition

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau[\Phi^{-1}(\tau)]^{\frac{1}{\alpha-1}}} = \infty \quad (4.5)$$

for some $\delta_0 > \tau_0 := \Phi(0)$, and φ satisfies the Calderón condition

$$\int_1^{\infty} \left(\frac{t}{\varphi(t)} \right)^{\frac{1}{n-2}} dt < \infty.$$

Let the set F have positive capacity for $\alpha = n$, and be an arbitrary closed set for $n-1 < \alpha < n$. If the family of all components of $D'_f \setminus E_*$ is equi-uniform over $f \in \mathfrak{DS}$ with respect to α -modulus, then every $f \in \mathfrak{DS}$ can be continuously extended to ∂D .

Furthermore, the corresponding extended family $\overline{\mathfrak{DS}}$ is equicontinuous in $\overline{D}$ with respect to the chordal metric h .

Proof. Conceptually, the proof of Theorem 4.7 is similar to the proof of Theorem 1.3. However, it is important to understand some differences between these proofs. First of all, as in the proof of Theorem 1.3, we will use some modulus estimate similar to (1.2). On the other hand, such an

estimate does not hold a priori in the class $\mathfrak{DS}$, so we must successively apply Lemmas 4.5 and 4.6. The second important difference is the use of the multiplicity $N(f, D)$ of the mapping, namely, if the estimate (1.2), used in the proof of Theorem 1.3, did not involve $N(f, D)$, then we must now take its participation into account in the context of Lemma 4.5. Finally, the third important difference is that estimates of the form (4.4) for the Orlicz-Sobolev classes are established only in those domains where the mapping is closed, see Lemma 4.6. Since our mappings, generally speaking, are not such, we should first reduce the study to this case.

Fix an arbitrary point $x_0 \in D$. According to Lemma 4.5, each mapping $f \in \mathfrak{DS}$ belongs to the class of lower Q -mappings relative to p -modulus at every point of the closure $\bar{D}$, where the majorant is given by

$$Q(x) = N \cdot K_{I, \alpha}^{\frac{p-n+1}{n-1}}(x, f)$$

with $\alpha = \frac{p}{p-n+1}$. From Lemma 4.3, we conclude that f satisfies the capacity estimate

$$\begin{aligned} \text{cap}_\alpha(f(\mathcal{E})) &\leq \int_{A(x_0, r_1, r_2)} N^{\frac{n-1}{p-n+1}} \cdot K_{I, \alpha}(x, f) \eta^\alpha(|x - x_0|) \, dm(x) \\ &\leq N^{\frac{n-1}{p-n+1}} \cdot \int_{A(x_0, r_1, r_2)} Q(x) \eta^\alpha(|x - x_0|) \, dm(x), \end{aligned}$$

$p = \frac{\alpha(n-1)}{\alpha-1}$, established for every Lebesgue measurable function

$$\eta: (r_1, r_2) \rightarrow [0, \infty]$$

that obeys the normalization condition $\int_{r_1}^{r_2} \eta(r) \, dr \geq 1$. The equicontinuity of the family $\mathfrak{DS}$ at any $x_0 \in D$ is then directly ensured by Lemma 4.4.

Next, let us establish the boundary equicontinuity. Specifically, we aim to show that for each finite boundary point $x_0 \in \partial D$ and any given $\varepsilon > 0$, one can find a radius $\delta = \delta(\varepsilon, x_0) > 0$ such that

$$h(f(x), f(y)) < \varepsilon \tag{4.6}$$

holds simultaneously for all $f \in \mathfrak{DS}$ and all points $x, y \in B(x_0, \delta) \cap D$.

$$h(f_k(x_k), f_k(x'_k)) \geq \frac{\mu_0}{2}. \tag{4.7}$$

By virtue of condition i_2 , there exists a system of contracting neighborhoods $V_k \subset B(x_0, 2^{-k})$, $k \in \mathbb{N}$, where each intersection $V_k \cap D$ is connected, and the set $(V_k \cap D) \setminus E$ splits into at most m components for some finite integer $m = m(x_0) \geq 1$.

Excluding the exceptional set E from our considerations in a manner identical to Lemma 2.3, we assume that $x_k, x'_k \notin E$. Lemma 2.1 then guarantees the existence of subsequences x_{k_j} and x'_{k_j} located within a specialized sequence of neighborhoods V_j around x_0 with $\text{diam } V_j \xrightarrow{j \rightarrow \infty} 0$. Crucially, each pair x_{k_j}, x'_{k_j} can be joined by a path γ_j lying inside $V_j \cap D$, such that γ_j intersects E at no more than $m - 1$ points. Retaining the original notation for these standard subsequences, we denote these joining paths by $\gamma_k: [0, 1] \rightarrow D$, where $\gamma_k(0) = x_k$ and $\gamma_k(1) = x'_k$ for $k \in \mathbb{N}$.

Consequently, the image path $f_k(\gamma_k)$ contains at most $m - 1$ points belonging to the target set E_* . Let these intersection points be represented as:

$$b_1 = f_k(\gamma_k(t_1)), \quad b_2 = f_k(\gamma_k(t_2)), \quad \dots \quad b_j = f_k(\gamma_k(t_j)),$$

$$t_0 := 0 \leq t_1 \leq t_2 \leq \dots \leq t_j \leq 1 = t_{j+1}, \quad 0 \leq j \leq m - 1.$$

Applying the triangle inequality to the metric relation (4.7), we obtain

$$\frac{\mu_0}{2} \leq h(f_k(x_k), f_k(x'_k)) \leq \sum_{r=0}^j h(f_k(\gamma_k(t_r)), f_k(\gamma_k(t_{r+1}))). \quad (4.8)$$

The inequality (4.8) implies that for each index k , there is an integer $1 \leq r = r(k) \leq m - 1$ satisfying

$$h(f_k(\gamma_k(t_{r(k)})), f_k(\gamma_k(t_{r(k)+1}))) \geq \frac{\mu_0}{2(j+1)} \geq \frac{\mu_0}{2m}. \quad (4.9)$$

The segment $G_k := |f_k(\gamma_k)|_{(t_{r(k)}, t_{r(k)+1})}$ lies entirely within the complementary domain $D' \setminus E_*$. By virtue of (4.9), G_k contains a subcontinuum $\tilde{G}_k$ whose chordal diameter satisfies $h(\tilde{G}_k) \geq \frac{\mu_0}{4m}$ uniformly for all $k \in \mathbb{N}$.

Since $\tilde{G}_k$ forms a continuum in a single component K_k of the open set $D'_{f_k} \setminus E_*$, we can exploit the equi-uniformity property for the pair of continua $\tilde{G}_k$ and K_{f_k} within K_k . Recalling that $h(K_{f_k}) \geq \delta$ from the definition of the mapping class, we use the positive number $\delta_* := \min\{\delta, \frac{\varepsilon}{4m}\}$ to conclude that there exists a uniform constant $C_0 > 0$ such that

$$M_\alpha(\Gamma(\tilde{G}_k, K_{f_k}, K_k)) \geq C_0 > 0, \quad k \in \mathbb{N}. \quad (4.10)$$

Our next goal is to show that the uniform lower bound (4.10) directly contradicts the integral constraints (1.4) and (4.5). Let Γ_k denote the family of half-open paths $\beta_k: [a, b) \rightarrow \mathbb{R}^n$ that start at $\tilde{G}_k$, belong to K_k , and terminate at the continuum K_{f_k} , so that $\lim_{t \rightarrow b-0} \beta_k(t) := B_k \in K_{f_k}$. It follows immediately from (4.10) that

$$M_\alpha(\Gamma_k) = M_\alpha(\Gamma(\tilde{G}_k, K_{f_k}, K_k)) \geq C_0 > 0, \quad k \in \mathbb{N}. \quad (4.11)$$

Adapting the spatial decomposition method from [4, Lemma 3.1], the open set $D \setminus E$ can be partitioned into a finite collection of components $D_1, \dots, D_s$, $1 \leq s < \infty$, such that the restriction of f_k to each D_i behaves as a closed mapping. We can identify a specific component, say D_1 , such that $f_k(D_1) = K$. By standard technical reductions, we assume that the restricted subpaths $\nabla_k := \gamma_k|_{[p_{s(k)}, q_{s(k)}]}$ are contained within D_1 , where the parameter intervals satisfy

$$t_{r(k)} < p_{s(k)} < q_{s(k)} < t_{r(k)+1},$$

while the endpoints converge as:

$$\begin{aligned} a_{s,k} &:= f_k(\gamma_k(p_s)) \xrightarrow{s \rightarrow \infty} w_k := f_k(\gamma_k(t_{r(k)})) \\ b_{s,k} &= f_k(\gamma_k(q_s)) \xrightarrow{s \rightarrow \infty} f_k(\gamma_k(t_{r(k)+1})). \end{aligned}$$

Proposition 2.2 guarantees the existence of a family Γ'_k consisting of maximal f_k -liftings $\alpha_k: [a, c) \rightarrow D$ of the family Γ_k of paths starting at $|\nabla_k|$.

Let us rule out the scenario where the liftings approach the boundary of the component, i.e., $\alpha_k \xrightarrow{k \rightarrow \infty} \partial D_1$. Seeking a contradiction, suppose that $\alpha_k(t) \xrightarrow{t \rightarrow c} \partial D_1$. Choosing a sequence $\varphi_m \in [0, c)$ with $\varphi_m \rightarrow c - 0$, the compactness of ∂D_1 allows us to locate points $w_m \in \partial D_1$ such that

$$h(\alpha_k(\varphi_m), \partial D_1) = h(\alpha_k(\varphi_m), w_m) \xrightarrow{m \rightarrow \infty} 0.$$

Extracting a convergent subsequence $w_m \rightarrow w_0 \in \partial D_1$, the triangle inequality yields

$$h(\alpha_k(\varphi_m), w_0) \leq h(\alpha_k(\varphi_m), w_m) + h(w_m, w_0) \xrightarrow{m \rightarrow \infty} 0. \quad (4.12)$$

This leads to two geometric alternatives: either $w_0 \in \partial D$ or $w_0 \in D$.

In the first scenario ($w_0 \in \partial D$), the mapping construction implies

$$f_k(\alpha_k(\varphi_m)) = \beta_k(\varphi_m) \xrightarrow{m \rightarrow \infty} \beta(c), \quad (4.13)$$

since the paths $\beta_k(t)$ belong to $K_k \subset D'_{f_k} \setminus E_*$. Combining (4.12) and (4.13), we obtain $\beta_k(c) \in C(f_k, \partial D) \subset E_*$, which is impossible since the entire path β_k lies in $D' \setminus E_*$, creating a clear topological contradiction.

In the second scenario ($w_0 \in D$), the point must belong to the exceptional set, so $w_0 \in E$. The limit relationship

$$f_k(\alpha_k(\varphi_m)) = \beta_k(\varphi_m) \rightarrow \beta_k(c)$$

then forces $\beta_k(c) \in f_k(E)$. Since $f_k^{-1}(E_*) = E$, this means $\beta_k(c) \in E_*$, which again violates the definition of the path family β_k . Thus, the condition $\alpha_k \rightarrow \partial D_1$ cannot hold.

Consequently, Proposition 2.2 implies that the liftings are complete, meaning $\alpha_k: [a, b] \rightarrow D$, with $\alpha_k \xrightarrow{t \rightarrow c-0} x_1 \in D_1$ and $f_k(\alpha_k(b)) = f_k(x_1)$. This implies that the terminal points satisfy $\alpha_k(b) \in f_k^{-1}(K_{f_k})$.

Invoking the definition of the class $\mathfrak{DS}_{\varphi, \Phi, \delta, \alpha}^{E_*, E, F, N, M}(D)$, we have the uniform separation property

$$h(f_k^{-1}(K_{f_k}), \partial D) \geq \delta > 0. \quad (4.14)$$

For a finite point $x_0 \neq \infty$, condition (4.14) guarantees that

$$f_k^{-1}(K_{f_k}) \subset D \setminus B(x_0, R_0) \quad (4.15)$$

holds for all indices k and some fixed $R_0 > 0$. Observe that $\Gamma_k > f_k(\Gamma'_k)$ and, therefore, $M_\alpha(\Gamma_k) \leq M_\alpha(f_k(\Gamma'_k))$ due to the minorization of the modulus. Choosing k large enough so that $2^{-k} < \mu_0$, and adjusting $R_0 < \mu_0$, we apply Lemmas 4.5 and 4.6 in combination with (4.15). For the exponent $\alpha := \frac{p}{p-n+1}$, this yields the modulus estimate

$$\begin{aligned} M_\alpha(\Gamma(\tilde{G}_k, K_{f_k}, K)) &\leq M_\alpha(f_k(\Gamma'_k)) \\ &\leq N^{\frac{n-1}{p-n+1}} \int_{A(x_0, 2^{-k}, R_0)} Q_{f_k}(x) \eta^\alpha(|x - x_0|) dm(x) \end{aligned} \quad (4.16)$$

valid for any nonnegative Lebesgue measurable function η that satisfies the boundary condition (1.3) with $\varepsilon = 2^{-k}$ and $\mu_0 = R_0$.

Let us analyze the spherical average integral

$$I_k = I(x_0, 2^{-k}, R_0) = \int_{2^{-k}}^{R_0} \frac{dr}{r^{\frac{n-1}{\alpha-1}} \tilde{q}_{k, x_0}^{\frac{1}{\alpha-1}}(r)},$$

where we define

$$\begin{aligned} \tilde{q}_{k, x_0}(r) &= \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(x_0, r)} \tilde{Q}_{f_k}(x) d\mathcal{H}^{n-1}, \\ \tilde{Q}_{f_k}(x) &= \begin{cases} Q_{f_k}(x), & Q(x) \geq 1 \\ 1, & Q(x) < 1. \end{cases} \end{aligned} \quad (4.17)$$

According to Lemma 2.3 and Remark 2.4, the integral diverges as the neighborhood shrinks:

$$I_k = \int_{2^{-k}}^{R_0} \frac{dr}{r^{\frac{n-1}{\alpha-1}} \tilde{q}_{k, x_0}^{\frac{1}{\alpha-1}}(r)} \xrightarrow{k \rightarrow \infty} \infty. \quad (4.18)$$

Given that $\tilde{q}_{k,x_0}(r) \geq 1$ almost everywhere, I_k remains finite and positive for all sufficiently large k . We put

$$\psi_k(t) = \begin{cases} \frac{1}{t^{\frac{n-1}{\alpha-1}} \tilde{q}_{k,x_0}^{\frac{1}{\alpha-1}}(t)}, & t \in (2^{-k}, R_0), \\ 0, & t \notin (2^{-k}, R_0). \end{cases}$$

Normalizing this function by setting $\eta_k(t) = \frac{\psi_k(t)}{I_k}$, we obtain a valid competitor for relation (1.3) on the interval $(2^{-k}, R_0)$. By applying condition (1.2), the structural inequality (4.16), the divergence property (4.18), and Fubini's theorem, we arrive at

$$\begin{aligned} M_\alpha(\Gamma_k) &\leq M_\alpha(f_k(\Gamma'_k)) \\ &\leq \frac{N^{\frac{n-1}{p-n+1}}}{I_k^\alpha} \int_{A(x_0, 2^{-k}, R_0)} \tilde{Q}_{f_k}(x) \cdot \psi_k^\alpha(|x - x_0|) \, dm(x) \\ &= \frac{N^{\alpha-1} \omega_{n-1}}{I_k^{\alpha-1}} \xrightarrow{k \rightarrow \infty} 0. \end{aligned} \tag{4.19}$$

It follows from (4.19) that

$$M_\alpha(\Gamma_k) \leq \Delta(k) \xrightarrow{k \rightarrow \infty} 0. \tag{4.20}$$

However, the asymptotic decay in (4.20) directly contradicts the uniform strictly positive lower bound established in (4.11). This contradiction completes the proof of the estimate (4.6).

Note that the uniform estimate (4.6) immediately guarantees that every mapping $f \in \mathfrak{DS}$ can be continuously extended to the boundary point x_0 . Indeed, if an extension was impossible, one could find sequences $x_k, y_k \rightarrow x_0$ such that $h(f(x_k), f(y_k)) \geq \mu_0$, directly violating (4.6).

To show that the extended family $\overline{\mathfrak{DS}}$ is equicontinuous at x_0 relative to the closure $\overline{D}$, we again suppose the contrary. This implies the existence of a point $x_0 \in \partial D$, a constant $\mu_0 > 0$, a sequence $x_k \in \overline{D}$ approaching x_0 , and mappings $f_k \in \overline{\mathfrak{DS}}$ such that

$$h(f_k(x_k), f_k(x_0)) \geq \mu_0. \tag{4.21}$$

Since the boundary extensions are continuous, we can assume without loss of generality that $x_k \in D$. Furthermore, relation (4.21) allows us to choose an auxiliary sequence $x'_k \in D$ that converges to x_0 and satisfies

$$h(f_k(x_k), f_k(x'_k)) \geq \frac{\mu_0}{2}.$$

This inequality directly contradicts the uniform boundary estimate (4.6), which fully settles the case for any finite point $x_0 \neq \infty$.

Finally, consider the case where $x_0 = \infty$. We introduce the auxiliary family of mappings $\tilde{f}_k := \psi \circ f_k$, where $\psi(y) = \frac{y}{|y|^2}$ denotes the standard inversion. By the hypotheses of Theorem 4.7, these transformed mappings satisfy the conditions (1.2)–(1.3) at the origin under the modified majorant

$$\tilde{Q}(x) := Q_{f_k}\left(\frac{x}{|x|^2}\right).$$

The conformal transformation $x = \frac{y}{|y|^2}$ generates the standard Jacobian $J(x, y) = \frac{1}{|y|^{2n}}$, meaning that relation (3.17) remains valid. Consequently, we can apply the exact analytical framework developed above to the function $Q_{f_k}(\frac{x}{|x|^2})$ instead of $Q_{f_k}(x)$. Replicating the arguments following equation (4.6) for the inverted family $\tilde{f}_k$, we arrive at an identical contradiction, which concludes the proof of Theorem 4.7. $\square$

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