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Interacting telegraph particles with mutual attraction in a Markov environment in the real line

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ABSTRACT
Publications on the study of the telegraph process are mainly devoted to various generalizations of this process either to multidimensional spaces or to various non-Markov environments in which this process evolves. Only a few works are devoted to the study of two or more telegraph processes that describe the evolution of interacting telegraph particles. In this paper, the characteristics of the time of the first collision of two particles with a shift toward each other are studied. We also study the number of collisions of n telegraph particles with a shift toward each other. Such a system can be an approximate model for the gravitational or electromagnetic interaction of two telegraph particles in a Markov environment.

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1. Introduction

In recent published works on clustering of particles on the real line, the authors are trying to model the attraction effect of gravitational forces, however the problems appear to be cumbersome, and some valuable asymptotic results are presented.[8,10,11,15–17] A typical assumption is that the position of particles are according to Poisson processes. In this paper we are dealing with the behavior of the random motion on only two particles driven by telegraph process with special relative velocities depending of an adjustable parameter Δ. We find a closed-form solution for the probability density function of the first-collision time of such particles, and some statistical moments. The number of collisions of n telegraph particles with a shift toward each other is estimated. This new stochastic environment can be considered as a generalization of the Arratia type flow for telegraph processes.[1,4,9,13] Furthermore, the collision of particles can be seen as the meeting time of them. So, it is related to first-passage problems, and to the difference of the two governing stochastic processes. In this sense, it is worth to mention the work in Kolesnik.[6] However, such an author does not consider any model for the mutual attraction between the particles, as we do in this work.

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Consider a Markov process $\{\xi(t), t \geq 0\}$ on the phase space $\{0, 1\}$ with a generator matrix

$$Q = \lambda \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The following equation

$$\frac{d}{dt} x(t) = v_{\xi(t)}, \quad v_1 > 0, \quad v_2 < 0, \quad x(0) = x_0, \tag{1.1}$$

determines the random evolution in the Markov environment $\xi(t)$. In the case, where $v_1 = -v_2$, and the process $x(t)$ is called the classic Goldstein-Kac telegraph process.[5,14]

In the sequel, the process $x(t)$ is said to be the telegraph process which represents the position of telegraph particle starting from x_0 with velocity switching process $\xi(t)$.

Let $\xi_i(t)$, $i = 1, 2$ be two independent Markov processes on the phase space $\{0, 1\}$ with the generator matrix Q.

Denote by $x_i(t)$ the position of telegraph particle i at a point of time $t \geq 0$. That is

$$\frac{d}{dt} x_i(t) = v^{(i)}_{\xi_i(t)}, \quad x_i(0) = x_i, \tag{1.2}$$

where $v^{(1)}_l = (-1)^l v + \Delta$, $v^{(2)}_l = (-1)^l v - \Delta$, $0 < \Delta < v$.

We should remark that $v^{(1)}_l$ is the velocity of particle 1, $v^{(2)}_l$ is the velocity of particle 2, and l is the step. It means that for $l = 0$ both particles are moving, where the velocity of the first particle is $v + \Delta$, whereas the second particle is moving at velocity $v - \Delta$, and if $l = 1$ the velocity of the first particle is $-v + \Delta$, and $-v - \Delta$ for the second one. This behavior is the key for attraction and eventually collision of both particles.

To be more precise, in the case $\xi_1(0) = 0$ and $\xi_2(0) = 1$ particle 1 starts moving with velocity $v^{(1)}_0 = v + \Delta$, and particle 2 starts moving with velocity $v^{(1)}_0 = -v - \Delta$. So, they move toward each other at the velocity $2v + 2\Delta$.

If say, $\xi_1(0) = 1$ and $\xi_2(0) = 1$, then particle 1 starts moving with velocity $v^{(1)}_0 = -v + \Delta$ and particle 2 starts moving with velocity $v^{(1)}_0 = -v - \Delta$, and they move toward each other at the velocity 2Δ. In addition, both particles move along the line at velocity $-v$, but this case is not considered here since we study the first collision of the particle in this work. The case $\xi_1(0) = 0$ and $\xi_2(0) = 0$ is much the same as the case $\xi_1(0) = 1$ and $\xi_2(0) = 1$, i.e. particles move toward each other at velocity 2Δ, and both particles move along the line at velocity v. In the case $\xi_1(0) = 1$ and $\xi_2(0) = 0$ particle 1 starts moving with velocity $v^{(1)}_0 = -v + \Delta$ and particle 2 starts moving with velocity $v^{(1)}_0 = v - \Delta$. Since $v > \Delta$, then both particles move away from each other at velocity $2v - 2\Delta$.

These three cases have initial velocities different from zero, so they might be modeling a warm gas. However, this is not exactly a gravitational attraction since

$\Delta = const$ and it is not $\Delta = at$ or something like that. Moreover, in our model, since for any $\Delta = const > 0$ the expectation of particles collision is finite, then we may confirm that the expectation of particles collision under gravitational attraction is also finite. The rate of attraction of both particles can be controlled with the parameter Δ.

We should mention that in case $\Delta = 0$ the expectation is infinite, and it is also infinite for two particles moving according to Wiener processes as well.

We assume that $z = x_2 - x_1 > 0$, and define $D(t) := x_2(t) - x_1(t)$.

Let us denote by $\eta(t) = (\xi_1(t), \xi_2(t))$ the two-dimensional process, and consider the following random variable

$$\tau_{(k_1,k_2)}(z) = \inf\{t \geq 0 : D(t) = 0,\ \eta(0) = (k_1, k_2)\}, \quad k_j \in \{0, 1\}. \tag{1.3}$$

It is easily seen that $\tau_{(k_1,k_2)}(z)$ is the instant of the first particles collision assuming $\eta(0) = (k_1, k_2)$.

2. Laplace transform of the first-time collision of two particles

Let us calculate the Laplace transforms of $\tau_{(k_1,k_2)}(z)$, $k_i \in \{0, 1\}$.

$$\varphi_{(k_1,k_2)}(s, z) = \mathbf{E}\left[e^{-s\tau_{(k_1,k_2)}(z)}\right],\ s > 0. \tag{2.1}$$

Lemma 2.1. For $t \geq \frac{z}{2v}$

$$\varphi_{(0,1)}(s, z) = \exp\left(\frac{2\lambda\Delta + s\Delta - \sqrt{v^2(s + 2\lambda)^2 - 4\lambda^2\left(v^2 - \Delta^2\right)}}{2\left(v^2 - \Delta^2\right)}\right) z, \tag{2.2}$$

$$\varphi_{(0,0)}(s, z) = \varphi_{(1,1)}(s, z) = \frac{2\lambda v + sv - \sqrt{v^2(s + 2\lambda)^2 - 4\lambda^2\left(v^2 - \Delta^2\right)}}{2\lambda(v - \Delta)} \tag{1}$$

$$\times \exp\left(\frac{2\lambda\Delta + s\Delta - \sqrt{v^2(s + 2\lambda)^2 - 4\lambda^2\left(v^2 - \Delta^2\right)}}{2\left(v^2 - \Delta^2\right)}\right) z, \tag{2.3}$$

and

$$\varphi_{(1,0)}(s, z) = \frac{(v + \Delta)\left(sv + 2\lambda v - \sqrt{v^2(s + 2\lambda)^2 - 4\lambda^2\left(v^2 - \Delta^2\right)}\right)}{(v - \Delta)\left(sv + 2\lambda v + \sqrt{v^2(s + 2\lambda)^2 - 4\lambda^2\left(v^2 - \Delta^2\right)}\right)} \tag{2}$$

$$\times \exp\left(\frac{2\lambda\Delta + s\Delta - \sqrt{v^2(s + 2\lambda)^2 - 4\lambda^2\left(v^2 - \Delta^2\right)}}{2\left(v^2 - \Delta^2\right)}\right) z. \tag{2.4}$$

Proof. The Laplace transforms of Equations (2.2)–(2.4) satisfy the system of integral equations

$$\begin{aligned}\varphi_{(0,1)}(s,z) &= e^{\frac{-s+2\lambda}{2v+2\Delta}z} + \frac{\lambda}{2v+2\Delta}\int_0^z e^{\frac{-s+2\lambda}{2v+2\Delta}u}\varphi_{(1,1)}(s,z-u)\,du \\ &\quad + \frac{\lambda}{2v}\int_0^z e^{\frac{-s+2\lambda}{2v+2\Delta}u}\varphi_{(0,0)}(s,z-u)\,du \\ &= e^{\frac{-s+2\lambda}{2v+2\Delta}z} + \frac{\lambda}{2v+2\Delta}e^{\frac{-s+2\lambda}{2v+2\Delta}z} \\ &\quad \int_0^z e^{\frac{s+2\lambda}{2v+2\Delta}u}\left(\varphi_{(1,1)}(s,u)+\varphi_{(0,0)}(s,u)\right)du, \end{aligned} \tag{2.5}$$

$$\begin{aligned}\varphi_{(0,0)}(s,z) &= e^{\frac{-s+2\lambda}{2\Delta}z} + \frac{\lambda}{2\Delta}\int_0^z e^{\frac{-s+2\lambda}{2\Delta}u}\varphi_{(1,1)}(s,z-u)\,du \\ &\quad + \frac{\lambda}{2\Delta}\int_0^z e^{\frac{-s+2\lambda}{2\Delta}u}\varphi_{(0,0)}(s,z-u)\,du \\ &= e^{\frac{-s+2\lambda}{2v+2\Delta}z} + \frac{\lambda}{2v}e^{\frac{-s+2\lambda}{2v+2\Delta}z} \\ &\quad \int_0^z e^{\frac{s+2\lambda}{2v+2\Delta}u}\left(\varphi_{(1,1)}(s,u)+\varphi_{(0,0)}(s,u)\right)du, \end{aligned} \tag{2.6}$$

$$\begin{aligned}\varphi_{(1,1)}(s,z) &= e^{\frac{-s+2\lambda}{2\Delta}z} + \frac{\lambda}{2\Delta}\int_0^z e^{\frac{-s+2\lambda}{2\Delta}u}\varphi_{(1,1)}(s,z-u)\,du \\ &\quad + \frac{\lambda}{2\Delta}\int_0^z e^{\frac{-s+2\lambda}{2\Delta}u}\varphi_{(0,0)}(s,z-u)\,du \\ &= e^{\frac{-s+2\lambda}{2v+2\Delta}z} + \frac{\lambda}{2v}e^{\frac{-s+2\lambda}{2v+2\Delta}z} \\ &\quad \int_0^z e^{\frac{s+2\lambda}{2v+2\Delta}u}\left(\varphi_{(1,1)}(s,u)+\varphi_{(0,0)}(s,u)\right)du. \end{aligned} \tag{2.7}$$

It is easily seen that $\varphi_{(0,0)}(s,z) = \varphi_{(1,1)}(s,z)$.

$$\begin{aligned}\varphi_{(1,0)}(s,z) &= \frac{\lambda}{2v-2\Delta}\int_0^\infty e^{\frac{-s+2\lambda}{2v-2\Delta}u}\left(\varphi_{(0,0)}(s,z+u)+\varphi_{(1,1)}(s,z+u)\right)du \\ &= \frac{\lambda}{2v-2\Delta}e^{\frac{s+2\lambda}{2v-2\Delta}z}\int_z^\infty e^{\frac{-s+2\lambda}{2v-2\Delta}u}\left(\varphi_{(0,0)}(s,u)+\varphi_{(1,1)}(s,u)\right)du. \end{aligned} \tag{2.8}$$

Differentiating Equations (2.5)–(2.8) with respect to z we obtain

$$\frac{\partial}{\partial z}\varphi_{(0,1)}(s,z) = \frac{-s+2\lambda}{2v+2\Delta}\varphi_{(0,1)}(s,z) + \frac{\lambda}{2v+2\Delta}\left(\varphi_{(0,0)}(s,z)+\varphi_{(1,1)}(s,z)\right) \tag{2.9}$$

$$\frac{\partial}{\partial z}\varphi_{(0,0)}(s,z) = \frac{-s+2\lambda}{2\Delta}\varphi_{(0,0)}(s,z) + \frac{\lambda}{2\Delta}\left(\varphi_{(0,1)}(s,z)+\varphi_{(1,0)}(s,z)\right) \tag{2.10}$$

$$\frac{\partial}{\partial z}\varphi_{(1,1)}(s,z)=\frac{-s+2\lambda}{2\Delta}\varphi_{(1,1)}(s,z)+\frac{\lambda}{2\Delta}\left(\varphi_{(0,1)}(s,z)+\varphi_{(1,0)}(s,z)\right) \tag{2.11}$$

$$\frac{\partial}{\partial z}\varphi_{(1,0)}(s,z)=\frac{s+2\lambda}{2v-2\Delta}\varphi_{(1,0)}(s,z)-\frac{\lambda}{2v-2\Delta}\left(\varphi_{(0,0)}(s,z)+\varphi_{(1,1)}(s,z)\right), \tag{2.12}$$

with the initial condition for all $i,\ j\in\{0,1\}$,

$$\varphi_{(i,j)}(0,z)=1,$$

$$\lim_{s\to+\infty}\varphi_{(i,j)}(s,z)=0,\ \ \forall z\geq 0, \tag{2.13}$$

and $\forall s\geq 0$.

$$\varphi_{(0,1)}(s,0)=1.$$

By solving system of Equations (2.9)–(2.12) with the conditions (2.13), we have

$$\varphi_{(0,1)}(s,z)=\exp\left(\frac{2\lambda\Delta+s\Delta-\sqrt{v^2(s+2\lambda)^2-4\lambda^2\left(v^2-\Delta^2\right)}}{2\left(v^2-\Delta^2\right)}z\right), \tag{2.14}$$

$$\varphi_{(0,0)}(s,z)=\varphi_{(1,1)}(s,z)=A(s)\exp\left(\frac{2\lambda\Delta+s\Delta-\sqrt{v^2(s+2\lambda)^2-4\lambda^2\left(v^2-\Delta^2\right)}}{2\left(v^2-\Delta^2\right)}z\right), \tag{2.15}$$

$$\varphi_{(1,0)}(s,z)=B(s)\exp\left(\frac{2\lambda\Delta+s\Delta-\sqrt{v^2(s+2\lambda)^2-4\lambda^2\left(v^2-\Delta^2\right)}}{2\left(v^2-\Delta^2\right)}z\right), \tag{2.16}$$

where

$$A(s)=\frac{2\lambda v+sv-\sqrt{v^2(s+2\lambda)^2-4\lambda^2\left(v^2-\Delta^2\right)}}{2\lambda(v-\Delta)} \tag{2.17}$$

and

$$B(s)=\frac{(v+\Delta)\left(sv+2\lambda v-\sqrt{v^2(s+2\lambda)^2-4\lambda^2\left(v^2-\Delta^2\right)}\right)}{(v-\Delta)\left(sv+2\lambda v+\sqrt{v^2(s+2\lambda)^2-4\lambda^2\left(v^2-\Delta^2\right)}\right)}. \tag{2.18}$$

3. Moments of the first-time collision of two particles

It is easily seen that

$$\mathbf{E}\left[\tau_{(0,1)}(z)\right] = \frac{-\partial}{\partial s}\varphi_{(0,1)}(s,z)\Big|_{s=0} = \frac{z}{2\Delta}. \tag{3.1}$$

Thus, we have finite expectation for $\Delta > 0$. It was studied in Pogorui[13] and Pogorui et al.,[14] the case of particle collision, where two particles move with constant absolute velocity. In our case, it corresponds to $\Delta = 0$. It was shown that $\mathbf{E}\left[\tau_{(0,1)}(z)\right] = +\infty$, which is in accordance with this result.

Then, it is easily verified that

$$\mathbf{E}\left[\tau_{(0,0)}(z)\right] = \mathbf{E}\left[\tau_{(1,1)}(z)\right] = \frac{-\partial}{\partial s}\varphi_{(0,0)}(s,z)\Big|_{s=0} = \frac{\lambda z + v}{2\lambda\Delta} \tag{3.2}$$

and

$$\mathbf{E}\left[\tau_{(1,0)}(z)\right] = \frac{-\partial}{\partial s}\varphi_{(1,0)}(s,z)\Big|_{s=0} = \frac{\lambda z + 2v}{2\lambda\Delta}. \tag{3.3}$$

From Equations (3.1)–(3.3) it is clear that as λ becomes larger, then the less these expectations differ from each other.

Now, let us calculate the second moments.

$$\mathbf{E}\left[\tau^2_{(0,1)}(z)\right] = \frac{\partial^2}{\partial s^2}\varphi_{(0,1)}(s,z)\Big|_{s=0} = \frac{\left(\lambda z\Delta + v^2\right)z}{4\lambda\Delta^3}. \tag{3.4}$$

Thus, unlike the first moment of $\tau_{(0,1)}(z)$, the second moment depends on velocity v.

Then,

$$\begin{aligned}\mathbf{E}\left[\tau^2_{(0,0)}(z)\right] = \mathbf{E}\left[\tau^2_{(1,1)}(z)\right] &= \frac{\partial^2}{\partial s^2}\varphi_{(1,1)}(s,z)\Big|_{s=0} \\ &= \frac{\Delta(\lambda z + v)^2 + v^2(\lambda z + v)}{4\lambda^2\Delta^3}\end{aligned} \tag{3.5}$$

and

$$\mathbf{E}\left[\tau^2_{(1,0)}(z)\right] = \frac{\partial^2}{\partial s^2}\varphi_{(1,0)}(s,z)\Big|_{s=0} = \frac{\Delta(\lambda z + 2v)^2 + v^2(\lambda z + 2v)}{4\lambda^2\Delta^3}. \tag{3.6}$$

Finally, we obtain the variances.

For $\tau_{(0,1)}(z)$:

$$\begin{aligned}\mathrm{Var}\left[\tau_{(0,1)}(z)\right] &= \mathbf{E}\left[\tau^2_{(0,1)}(z)\right] - \left(\mathbf{E}\left[\tau_{(0,1)}(z)\right]\right)^2 \\ &= \frac{(\lambda z\Delta + v^2)z}{4\lambda\Delta^3} - \frac{z^2}{4\Delta^2} = \frac{v^2 z}{4\lambda\Delta^3}.\end{aligned}$$

For $\tau_{(0,0)}(z)$ and $\tau_{(1,1)}(z)$:

$$\mathrm{Var}\left[\tau_{(0,0)}(z)\right] = \mathrm{Var}\left[\tau_{(1,1)}(z)\right] = \frac{\Delta(\lambda z + v)^2 + v^2(\lambda z + v)}{4\lambda^2\Delta^3} - \frac{(\lambda z + v)^2}{4\lambda^2\Delta^2}$$

$$= \frac{v^2(\lambda z + v)}{4\lambda^2\Delta^3}.$$

For $\tau_{(1,0)}(z)$:

$$\mathrm{Var}\left[\tau_{(1,0)}(z)\right] = \frac{\Delta(\lambda z + 2v)^2 + v^2(\lambda z + 2v)}{4\lambda^2\Delta^3} - \frac{(\lambda z + 2v)^2}{4\lambda^2\Delta^2} = \frac{v^2(\lambda z + 2v)}{4\lambda^2\Delta^3}.$$

Remark **(Kac's condition).** It follows from the obtained expressions for the variances that

$$\mathrm{Var}\left[\tau_{(k,l)}(z)\right] \sim \frac{v^2}{\lambda}, \quad (k,l) \in \{0,1\},$$

up to multiplicative factors depending on z.

Consider the limiting regime where $\lambda \to \infty$ and $v \to \infty$ in such a way that

$$\frac{\lambda}{v^2} \to \sigma^2 > 0.$$

This scaling is known as the *Kac's condition*. In this case, the variances remain finite and satisfy

$$\mathrm{Var}\left[\tau_{(k,l)}(z)\right] \to \frac{z}{4\sigma^2\Delta^3},$$

up to constants depending on the initial state.

This regime corresponds to the classical diffusion limit of the telegraph process.

4. Distribution of the first collision of two particles

Denote by $f_{(k_1,k_2)}(t,z)\,dt = P\left(\tau_{(k_1,k_2)}(z) \in dt\right)$ the probability density function (pdf) of $\tau_{(k_1,k_2)}(z)$. Now, our purpose is to calculate the pdf of the random variable $\tau_{(k_1,k_2)}(z)$.

In order to obtain $f_{(0,1)}(t,z)$ we should find the inverse Laplace of $\varphi_{(0,1)}(s,z)$, i.e.

$$f_{(0,1)}(t,z) = \mathcal{L}^{-1}\left(e^{\frac{\Delta(s+2\lambda)-\sqrt{v^2(s+2\lambda)^2+4\lambda^2\left(\Delta^2-v^2\right)}}{2\left(v^2-\Delta^2\right)}z}, s\right).$$

It is well-known that

$$\mathcal{L}^{-1}\left(e^{\frac{\Delta(s+2\lambda)-\sqrt{v^2(s+2\lambda)^2+4\lambda^2\left(\Delta^2-v^2\right)}}{2\left(v^2-\Delta^2\right)}z}, s\right) = e^{-2\lambda t}\mathcal{L}^{-1}\left(e^{\frac{\Delta s-\sqrt{v^2s^2+4\lambda^2\left(\Delta^2-v^2\right)}}{2\left(v^2-\Delta^2\right)}z}, s\right). \tag{4.1}$$

So, we have to obtain

$$\mathcal{L}^{-1}\left(e^{\frac{\Delta s-\sqrt{v^2s^2+4\lambda^2\left(\Delta^2-v^2\right)}}{2\left(v^2-\Delta^2\right)}z}, s\right). \tag{4.2}$$

After some algebraic modifications we have the equivalent expression:

$$\mathcal{L}^{-1}\left(e^{\frac{\Delta s-\sqrt{v^2s^2+4\lambda^2\left(\Delta^2-v^2\right)}}{2\left(v^2-\Delta^2\right)}z},s\right)=\mathcal{L}^{-1}\left(e^{-Kzs}e^{-Rz\sqrt{(s-T)(s+T)}},s\right), \quad (4.3)$$

where

$$K=\frac{1}{2\Delta\left(1-v^2/\Delta^2\right)},$$

$$T=4\lambda^2\left(1-\Delta^2/v^2\right),$$

$$R=\frac{1}{2v\left(1-\Delta^2/v^2\right)}.$$

Then, the inverse Laplace transform of

$$e^{-Kzs}e^{-Rz\sqrt{(s-T)(s+T)}}$$

with respect to s yields the $f(t,z)$ (see as the basic Laplace transform,[7, p.239, Formula 93] for $\mathcal{L}^{-1}\left(e^{-ks}e^{-k\sqrt{s^2-a^2}},s\right)$). So, $f_{(0,1)}(t,z)=e^{-2\lambda t}f(t,z)$. Hence,

$$f_{(0,1)}(t,z)=\frac{RTe^{-2\lambda t}z}{\sqrt{(t-Kz)^2-(Rz)^2}}I_1\left(T\sqrt{(t-Kz)^2-(Rz)^2}\right) \quad (4.4)$$

for $t\geq(K+R)z$, and zero otherwise. The function $I_1(\cdot)$ is the modified Bessel function of order one.

Some recent efforts have been done for an efficient simulation of these complex problems,[3] and this pdf closed-form might be useful for a better understanding of these issues.

5. Estimating the number of collisions of telegraph particles

The following model of an ideal gas with finite molecular velocities is considered:

(1) For the set of real numbers $y_1<y_2<\cdots<y_n$ consider a set of independent telegraph processes $S_i(t)$, $i=1,2,\ldots,n$, $t\geq0$, $S_i(0)=y_i$, with the same parameters v and λ.
(2) Let the process $x(y_i,t)$, $t\geq0$, which describes the position of particle i at time t, and it coincides with ordinal statistics $S_{(i)}(t)$ of processes $S_i(t)$, $i=1,2,\ldots,n$:

$$x(y_i,t)=S_{(i)}(t).$$

This method of defining particle motion corresponds to each particle in the system moving according to its corresponding telegraph process until it collides with another particle. A hard collision between two particles is understood as a

collision in which the particles reverse their direction of motion, that is, exchange momentum. In process terms, this means that the particles exchange telegraph processes describing their motion.

Let us denote by $C_{(k,l)}^{(i,j)}(t,z)$ the number of collisions of particles i and j (or $x(y_i,t)$ and $x\left(y_j,t\right)$ in processes terms) during time $(0,t)$, $t>0$ assuming $\eta(0)=(k,l)$, and it is equal to the number of intersections of telegraph processes $S_i(t)$ and $S_j(t)$ assuming $\left(\xi_i(0),\xi_j(0)\right)=(k,l)$. Consider the renewal function $H_{(k,l)}^{(i,j)}(t,z)=\mathbf{E}\left[C_{(k,l)}^{(i,j)}(t,z)\right]$. It is well–known from renewal theory that the renewal function

$$H_{(1,0)}^{(i,j)}(t,z)=\frac{2\Delta\lambda}{\lambda z+v}t+o(t),$$

$$H_{(0,0)}^{(i,j)}(t,z)=H_{(1,1)}^{(i,j)}(t,z)=\frac{2\Delta\lambda}{\lambda z+2v}t+o(t), \tag{5.1}$$

$$H_{(0,1)}^{(i,j)}(t,z)=\frac{2\Delta}{z}t+o(t).$$

It is known[13] that if $\Delta=0$ then

$$H_{(k,l)}^{(i,j)}(t,z)=C\sqrt{t}+o\left(\sqrt{t}\right),$$

where $C=$ constant.

Denote by $C_{(k_1,k_2,\ldots,k_n)}(t,y_1,y_2,\ldots,y_n)$, $k_i\in\{0,1\}$, $y_1<y_2<\cdots<y_n$ the number of collisions of particles $x(y_i,t)$, $i=1,2,\ldots,n$ during time $(0,t)$, $t>0$, assuming

$$\eta(0)=(\xi_1(0),\xi_2(0),\ldots,\xi_n(0))=(k_1,k_2,\ldots,k_n).$$

It is easily seen that

$$\begin{aligned}H_{(k_1,k_2,\ldots,k_n)}(t,y_1,y_2,\ldots,y_n)&=\mathbf{E}\left[C_{(k_1,k_2,\ldots,k_n)}(t,y_1,y_2,\ldots,y_n)\right]\\&=\sum_{i=1}^{n-1}H_{(k_i,k_{i+1})}(t,y_{i+1}-y_i).\end{aligned}$$

Taking into account (5.1) we have

$$H_{(k_1,k_2,\ldots,k_n)}(t,y_1,y_2,\ldots,y_n)=n(n-1)Ct+o(t),$$

where C is a constant depending on the initial positions of the particles and their initial velocities.

6. Conclusions and further work

The interaction of two particles governed by telegraph processes can be seen as the meeting time of such stochastic processes. So, this problem is somehow related to first passage times, or the difference of two telegraph processes when

the difference is zero. However, in this work we have considered mutual attraction of such particles that add complexity to the model. This mutual attraction is motivated by physical phenomena such as gravitation, electromagnetic fields, strong and weak fields at the atomic level. Under this model we have calculated the variance of the first-time collision of two telegraph particles as well as the probability density function. In addition, we provide an estimate for the number of collisions of such particles. We should mention that there are some recent interesting works dealing with the maximum or supremum of the first meeting time of telegraph processes,[2,12] and it might be a challenging problem to study such a supremum under the interaction of particles with mutual attraction. We are also aware that more sophisticated mutual attraction schemes may need to be developed to capture more realistic situations.

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Author contributions

CRediT: **Anatoliy A. Pogorui**: Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing; **Ramón M. Rodríguez-Dagnino**: Formal analysis, Methodology, Writing – original draft, Writing – review & editing.

Disclosure statement

No.

Data availability statement

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