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# On Koebe's theorem for mappings with integral constraints

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We study mappings that satisfy the inverse modulus inequality of Poletsky type with respect to $p$-modulus. Given $n-1 < p \leqslant n$, we show that the image of some ball contains a ball of a fixed radius uniformly under some family of mappings. This statement can be interpreted as an analogue of the well-known Koebe's theorem for analytic functions, but in more general classes. As a consequence, we obtain the openness and discreteness of the limit mapping in the class under study. The paper also separately studies mappings of the Orlicz-Sobolev classes for which an analogue of the Koebe one-quarter theorem is obtained.

*Key words and phrases:* quasiconformal mapping, mapping with finite distortion, Orlicz-Sobolev mapping, Koebe theorem.

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## 1 Introduction

The classical Koebe theorem for analytic functions states that conformal mappings of the unit disk with hydrodynamic normalization at the origin cover a disk centered at the origin of radius $1/4$. Let us directly recall the formulation of this theorem, see [4, Theorem 1.3].

**Theorem A.** *Let $f : \mathbb{D} \to \mathbb{C}$ be an univalent analytic function such that $f(0) = 0$ and $f'(0) = 1$. Then the image of $f$ covers the open disk centered at $0$ of radius one-quarter, that is, $f(\mathbb{D}) \supset B(0, 1/4)$.*

Many authors have obtained some other versions of the analogue of Koebe's theorem, including results for wider classes of mappings (see, e.g., [1, 12, 15]). In particular, we have recently obtained results on this topic (see [22]). Also in [5], some results on the convergence of mappings were obtained, which essentially used Koebe's theorem, established for more general classes of mappings compared to [22]. As for the present paper, we consider here classes of mappings satisfying the inverse Poletsky inequality, in which the majorant $Q$ may depend on the mapping. The so-called integral constraints are imposed on these majorants, which control them by means of a certain convex increasing function $\Phi$ (see, e.g., [17]). The main result of the paper is an analogue of Koebe's theorem for such mappings. As a consequence, we also establish the openness and discreteness of the limit mapping using the approaches we have

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developed, as well as the methodology of the paper [5]. One of the goals of the paper is also to obtain corresponding results for Orlicz-Sobolev classes. We establish these results using the connection between Orlicz-Sobolev classes and mappings with the inverse Poletsky inequality.

Below $dm(x)$ denotes the element of the Lebesgue measure in $\mathbb{R}^n$. Throughout the paper, the boundary $\partial A$ and the closure $\overline{A}$ of a set $A$ should be understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Recall that, a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called *admissible* for the family $\Gamma$ of paths $\gamma$ in $\mathbb{R}^n$, if the relation $\int_\gamma \rho(x)\,|dx| \geqslant 1$ holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write $\rho \in \operatorname{adm}\Gamma$. Given $p \geqslant 1$, *p-modulus* of $\Gamma$ is defined by the equality $M_p(\Gamma) = \inf_{\rho\in \operatorname{adm}\Gamma} \int_{\mathbb{R}^n} \rho^p(x)\,dm(x)$. Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0, r_1, r_2) = \left\{y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2\right\}.$$

Given $x_0 \in \mathbb{R}^n$, we put $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, $\mathbb{B}^n = B(0, 1)$. A mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$.

Given sets $E$, $F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$, and $0 < r_1 < r_2 < r_0 < \infty$, we denote by $\Gamma_f(y_0, r_1, r_2)$ a family of all paths $\gamma$ in $D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function and let $p \geqslant 1$. We say that $f$ *satisfies the inverse Poletsky inequality at a point* $y_0 \in \overline{f(D)} \setminus \{\infty\}$ *with respect to p-modulus,* if the relation

$$M_p(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y) \cdot \eta^{\,p}(|y - y_0|)\,dm(y) \tag{1}$$

holds for any $0 < r_1 < r_2 < r_0 < \infty$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1\,. \tag{2}$$

The definition of the relation (1) at the point $y_0 = \infty$ may be given by the using of the inversion $\psi(y) = \frac{y}{|y|^2}$ at the origin. This means that $f$ satisfies the relation (1) for $y_0 = 0$ and $\widetilde{Q}(y) := Q\left(\frac{y}{|y|^2}\right)$ instead $Q(y)$.

Quite a lot is known about the role of relations (1)–(2) in mapping theory, as well as about the fulfillment of these conditions in specific classes (see, e.g., [11, 22, 24, 25]). We also indicate the application of these inequalities to the Orlicz-Sobolev classes, see the last section of the manuscript.

Let $h$ be a chordal metric in $\overline{\mathbb{R}^n}$, defined by

$$h(x,\infty) = \frac{1}{\sqrt{1+|x|^2}} \qquad \text{and} \qquad h(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \quad x \neq \infty \neq y\,.$$

Let $h(E) := \sup\limits_{x,y\in E} h(x, y)$ be a chordal diameter of a set $E \subset \overline{\mathbb{R}^n}$ (see, e.g., [25, Definition 12.1]).

Given $p \geqslant 1$, a non-decreasing function $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$, $a, b \in D$, $a \neq b$, $\delta > 0$, we denote by $\mathfrak{F}^{\Phi,p}_{a,b,\delta}(D)$ the family of all open discrete mappings $f : D \to \mathbb{R}^n$, $n \geqslant 2$, for which there exists a

Lebesgue measurable function $Q = Q_f : \mathbb{R}^n \to [0,\infty]$ satisfying relations (1)–(2) at any point $y_0 \in \overline{\mathbb{R}^n}$ with

$$\int\limits_{\mathbb{R}^n} \Phi(Q_f(y)) \cdot \frac{dm(y)}{(1+|y|^2)^n} < \infty \tag{3}$$

such that $h(f(a), f(b)) \geqslant \delta$. The following statement holds.

**Theorem 1.** *Let $n \geqslant 2$, $p \in (n-1, n]$ and let $D$ be a domain in $\mathbb{R}^n$. Let also $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ be an increasing convex function that satisfies the condition*

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau(\Phi^{-1}(\tau))^{\frac{1}{p-1}}} = \infty \tag{4}$$

*for some $\delta > \Phi(0)$. Assume that the family $\mathfrak{F}^{Q,p}_{a,b,\delta}(D)$ is equicontinuous at $a$ and $b$. Then for every compactum $K$ in $D$ and for every $0 < \varepsilon < \operatorname{dist}(K, \partial D)$ there exists $r_0 = r_0(\varepsilon, K) > 0$, which does not depend on $f$, such that $f(B(x_0, \varepsilon)) \supset B_h(f(x_0), r_0)$ for all $f \in \mathfrak{F}^{\Phi,p}_{a,b,\delta}(D)$ and all $x_0 \in K$, where $B_h(f(x_0), r_0) = \{w \in \overline{\mathbb{R}^n} : h(w, f(x_0)) < r_0\}$.*

**Remark 1.** *Assume that, $Q$ satisfies* (3) *with some $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$. Set*

$$\widetilde{Q}(y) = \begin{cases} Q(y), & Q(y) \geqslant 1, \\ 1, & Q(y) < 1. \end{cases}$$

*Observe that $\widetilde{Q}(y)$ satisfies* (3)*. Indeed,*

$$\begin{aligned} \int\limits_D \Phi(\widetilde{Q}(y)) \frac{dm(y)}{(1+|y|^2)^n} &= \int\limits_{\{y\in D: Q(y)<1\}} \Phi(\widetilde{Q}(y)) \frac{dm(y)}{(1+|y|^2)^n} \\ &\quad + \int\limits_{\{y\in D: Q(y)\geqslant 1\}} \Phi(\widetilde{Q}(y)) \frac{dm(y)}{(1+|y|^2)^n} \\ &\leqslant M_0 + \Phi(1) \int\limits_{\mathbb{R}^n} \frac{dm(y)}{(1+|y|^2)^n} = M_0' < \infty\,. \end{aligned}$$

**Remark 2.** *If $p = n$ and $Q \in L^1(\mathbb{R}^n)$, then the family $\mathfrak{F}^{Q,p}_{a,b,\delta}(D)$ is equicontinuous. Indeed, let $\mathfrak{F}_Q(D)$ be a family of all open discrete mappings in $D$ such that the relations* (1)–(2) *holds for any $y_0 \in f(D)$ with $p = n$. If $Q \in L^1(\mathbb{R}^n)$, then for any $x_0 \in D$ and any $r_0 > 0$ such that $0 < r_0 < \operatorname{dist}(x_0, \partial D)$ the inequality $|f(x) - f(x_0)| \leqslant \frac{C_n \cdot (\|Q\|_1)^{1/n}}{\log^{1/n}\left(1+\frac{r_0}{2|x-x_0|}\right)}$ holds for any $x, y \in B(x_0, r_0)$ and $f \in \mathfrak{F}_Q(D)$, where $\|Q\|_1$ denotes the $L^1$-norm of $Q$ in $\mathbb{R}^n$ and $C_n > 0$ is some constant depending only on $n$. In particular, $\mathfrak{F}_Q(D)$ is equicontinuous in $D$ (see, e.g., [21, Theorem 1.1]). Thus, Theorem 1 holds for $\mathfrak{F}^{Q,n}_{a,b,\delta}(D)$ under the condition $Q \in L^1(\mathbb{R}^n)$ (see, e.g., [22]).*

Note that the above analogue of Koebe's theorem has an important application in the field of convergence of mappings. In this connection, we first recall the classical result on the convergence of quasiregular mappings.

**Theorem B.** *Let $f_j : D \to \mathbb{R}^n$, $n \geqslant 2$, $j = 1, 2, \ldots$, be a sequence of $K$-quasiregular mappings converging to some mapping $f : D \to \mathbb{R}^n$ as $j \to \infty$ locally uniformly in $D$. Then either $f$ is $K$-quasiregular, of $f$ is a constant. In particular, in the first case $f$ is discrete and open (see [16, Theorems 8.6.VI]).*

As for the classes we are studying in (1)–(2), we have the following analogue of Theorem B (cf. [5, Theorem 1.1]).

**Theorem 2.** *Let $D$ be a domain in $\mathbb{R}^n$, $n \geqslant 2$, $p > n-1$ and $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ be a non-decreasing function. Let $f_j : D \to \mathbb{R}^n$, $n \geqslant 2$, $j = 1,2,\ldots$, be a sequence of open discrete mappings satisfying the conditions* (1)–(2) *with some function $Q = Q_j : \mathbb{R}^n \to [0,\infty]$ at any point $y_0 \in \overline{\mathbb{R}^n}$ for which* (3) *holds with $Q_{f_j} := Q_j$ and converging to some mapping $f : D \to \mathbb{R}^n$ as $j \to \infty$ locally uniformly in $D$. Assume that the condition* (4) *holds. Then either $f$ is a constant, or $f$ is open and discrete.*

**Remark 3.** *For the case $p = n$ and some another conditions on the function $Q$, Theorem 2 was established in [22], except for the assertion about the discreteness of the limit mapping. In [5], the mentioned discreteness was proved even for the case of a non-conformal modulus. Our contribution to this result is that we do not require the family of mappings in* (1)–(2) *to have a fixed majorant $Q$. The proof of openness is similar to [22], the methodology of proving discreteness is from [5].*

## 2 Preliminaries

Let $(X,\mu)$ be a metric space with measure $\mu$. For each real number $n \geq 1$, we define *the Loewner function* $\phi_n : (0,\infty) \to [0,\infty)$ on $X$ as $\phi_n(t) = \inf\{M_n(\Gamma(E,F,X)) : \Delta(E,F) \leqslant t\}$, where the infimum is taken over all disjoint nondegenerate continua $E$ and $F$ in $X$ and $\Delta(E,F) := \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}$. A pathwise connected metric measure space $(X,\mu)$ is said to be a *Loewner space* of exponent $n$, or an $n$-Loewner space, if the Loewner function $\phi_n(t)$ is positive for all $t > 0$ (see [11, Section 2.5] or [7, Ch. 8]). Observe that $\mathbb{R}^n$ and $\mathbb{B}^n \subset \mathbb{R}^n$ are Loewner spaces (see [7, Theorem 8.2 and Example 8.24(a)]). As known, the condition $\mu(B(x_0,r)) \geqslant C \cdot r^n$ holds in Loewner spaces $X$ for a constant $C > 0$, every point $x_0 \in X$ and all $r < \operatorname{diam} X$. The following definition can be found in [7, Sec. 1.4, Ch. I] or [2, Sec. 1]. A measure $\mu$ in a metric space is called doubling if all balls have finite and positive measure and there is a constant $C \geqslant 1$ such that $\mu(B(x_0,2r)) \leq C \cdot \mu(B(x_0,r))$ for every $x_0 \in X$ and all $r > 0$. We also call a metric measure space $(X,\mu)$ *doubling* if $\mu$ is a doubling measure. A metric space $(X,d,\mu)$ is called $\widetilde{Q}$*-Ahlfors-regular* for some $\widetilde{Q} \geqslant 1$ if for any $x_0 \in X$ and some constant $C \geqslant 1$ we have $\frac{1}{C}R^{\widetilde{Q}} \leqslant \mu(B(x_0,R)) \leqslant CR^{\widetilde{Q}}$. As is well known, Ahlfors $\alpha$-regular spaces have Hausdorff dimension $\alpha$ (see [7, p. 61–62]). Let $(X,d,\mu)$ be a metric measure space with metric $d$ and a locally finite Borel measure $\mu$. Following [7, §7.22], a Borel function $\rho\colon X \to [0,\infty]$ is said to be an *upper gradient* of a function $u\colon X \to \mathbb{R}$ if $|u(x) - u(y)| \leqslant \int_\gamma \rho\,|dx|$ for any rectifiable path $\gamma$ connecting the points $x$ and $y \in X$, where, as usual, $\int_\gamma \rho\,ds$ denotes the linear integral of the function $\rho$ over the path $\gamma$. We say that a space $X$ admits the $(1;p)$-Poincaré inequality if there exist constants $C \geqslant 1$ and $\tau > 0$ such that

$$\frac{1}{\mu(B)}\int_B |u-u_B|\,d\mu(x) \leqslant C(\operatorname{diam} B)\left(\frac{1}{\mu(\tau B)}\int_{\tau B}\rho^p\,d\mu(x)\right)^{1/p}$$

for any ball $B \subset X$ and arbitrary locally bounded continuous function $u\colon X \to \mathbb{R}$ and any upper gradient $\rho$ of $u$, where $u_B := \frac{1}{\mu(B)}\int_B u\,d\mu(x)$.

The following result holds (see [2, Proposition 4.7]).

**Proposition 1.** *Let $X$ be a $Q$-Ahlfors regular metric measure space that supports $(1;p)$-Poincaré inequality for some $p > 1$ such that $Q - 1 < p \leqslant Q$. Then there exists a constant $M > 0$ having the property that for $x \in X$, $R > 0$ and continua $E$ and $F$ in $B(x,R)$, we have*

$$M_p(\Gamma(E,F,X)) \geqslant \frac{1}{M} \cdot \frac{\min\{\operatorname{diam} E, \operatorname{diam} F\}}{R^{1+p-Q}}.$$

**Lemma 1.** *Let $n \geqslant 2$, $p \in (n-1,n]$, $x_0 \in \mathbb{R}^n$, $\varepsilon_1 > 0$, and $A$ be a (non-degenerate) continuum in $B(x_0,\varepsilon_1) \subset \mathbb{R}^n$. Let $r > 0$ and $C_j$, $j = 1,2,\ldots$, be a sequence of continua in $B(x_0,\varepsilon_1)$ such that $h(C_j) \geqslant r$, $h(C_j) = \sup_{x,y\in C_j} h(x,y)$. Then there is $R_0 > 0$ such that $M_p(\Gamma(C_j,A,D)) \geqslant R_0$ for all $j \in \mathbb{N}$.*

*Proof.* By comments given above, the unit ball $\mathbb{B}^n$ is Ahlfors $n$-regular. Now, the ball $B(x_0,\varepsilon_1)$ is also is Ahlfors $n$-regular. In addition, $(1;p)$-Poincaré inequality holds for any $p \geqslant 1$ (see [6, Theorem 10.5]). Now, the desired statement follows by Proposition 1. □

Let $E_0$, $E_1$ be sets in $D \subset \mathbb{R}^n$. The following estimate holds (see [3, Theorem 4]).

**Proposition 2.** *Let $A(0,a,b) = \{a < |x| < b\}$ be a ring containing in $D \subset \mathbb{R}^n$ such that $S(0,r)$ intersects $E_0$ and $E_1$ for any $r \in (a,b)$, where $E_0 \cap E_1 = \varnothing$. Then*

$$M_p(\Gamma(E_0,E_1,D)) \geqslant \frac{2^n b_{n,p}}{n-p}(b^{n-p} - a^{n-p})$$

*for any $p \in (n-1,n)$, where $b_{n,p}$ is a constant depending only $n$ and $p$.*

The version of the following lemma is established in the case $p = n$ in [14, Theorem 3.1]. For the case $p \neq n$ its proof is similar and, therefore, is omitted.

**Lemma 2.** *Let $p > n-1$, $F_1$, $F_2$, $F_3$ be three sets in a domain $D$ and let $\Gamma_{i,j} = \Gamma(F_i,F_j,D)$, $1 \leqslant i,j \leqslant 3$. Then*

$$M_p(\Gamma_{1,2}) \geqslant 3^{-p} \min\{M_p(\Gamma_{1,2}), M_p(\Gamma_{2,3}), \inf M_p(\Gamma(|\gamma_{1,3}|,|\gamma_{2,3}|,D))\},$$

*where the infimum is taken over all rectifiable paths $\gamma_{1,3} \in \Gamma_{1,3}$ and $\gamma_{2,3} \in \Gamma_{2,3}$.*

A version of the following lemma is established in the case $p = n$ in [14, Theorem 3.3]. We are interested in the case $p \neq n$.

**Lemma 3.** *Let $p > n-1$, and $F_1$, $F_2$, $F_3$ be three sets in a domain $D$. Let $D$ contain the spherical ring $A(x_0,a,b)$, $x_0 \in \mathbb{R}^n$, $0 < a < b < \infty$, and $F_3$ lies in $\overline{B(x_0,a)}$. Let $\Gamma_{ij}$ be as in Lemma 2. If one of the following three conditions is satisfied:*

*(a) $F_i$ lies in $\mathbb{R}^n \setminus B(x_0,b)$, $i = 1,2$;*

*(b) $F_1$ lies in $\mathbb{R}^n \setminus B(x_0,b)$, and $F_2$ is connected with $d(F_2) > 2b$;*

*(c) $F_i$ is connected with $d(F_i) > 2b$, $i = 1,2$,*

*then, whenever $p = n$,*

$$M(\Gamma_{1,2}) > \min\left\{M(\Gamma_{1,3}), M(\Gamma_{2,3}), c_n \log\frac{b}{a}\right\},$$

*where $c_n$ is a positive constant depending only on $n$. If $p \neq n$, we have*

$$M_p(\Gamma_{1,2}) > \min\left\{M_p(\Gamma_{1,3}), M_p(\Gamma_{2,3}), \frac{2^n b_{n,p}}{n-p}(b^{n-p} - a^{n-p})\right\}. \tag{5}$$

*Proof.* The case $p = n$ is discussed in detail in [14, Theorem 3.3]. We should consider the case $p \neq n$. We may assume that $F_1$, $F_2$, $F_3$ are nonempty sets. If *(a)* is satisfied, then the assertion follows directly from Lemma 2 and Proposition 2. Assume next that *(b)* or *(c)* is satisfied. Choose $\rho \in \operatorname{adm}\Gamma_{1,2}$. If at least one of the conditions $\int_{\gamma_{1,3}} \rho\,|dx| \geqslant 1/3$, $\int_{\gamma_{2,3}} \rho\,|dx| \geqslant 1/3$ holds for every rectifiable path $\gamma_{1,3} \in \Gamma_{1,3}$ or $\gamma_{2,3} \in \Gamma_{2,3}$, respectively, then inequality (5) holds. Otherwise, $\int_{\alpha} \rho\,|dx| \geqslant 1/3$ holds for every rectifiable path $\alpha \in \Gamma(F_1 \cup |\gamma_{1,3}|, F_2 \cup |\gamma_{2,3}|, D)$. Therefore, since $S(x_0, t)$ meets both $F_1 \cup |\gamma_{1,3}|$ and $F_2 \cup |\gamma_{2,3}|$ for $a < t < b$ and since $D$ contains the spherical ring $A(x_0, a, b)$, we obtain by Proposition 2 that $M_p(\Gamma_{1,2}) \geqslant \frac{2^n b_{n,p}}{n-p}(b^{n-p} - a^{n-p})$. Thus, the relation (5) holds, as required. Lemma is proved. □

Finally, we have the following assertion (see [20, Theorem 1.1], cf. [13, Lemma 1.15]).

**Proposition 3.** *Let $D$ be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $n-1 < p \leqslant n$. If $A$ and $A^*$ are (nondegenerate) continua in $D$, then $M_p(\Gamma(A, A^*, D)) > 0$.*

## 3 Main Lemmas

The following statement generalizes Lemma 1 for the case of arbitrary continuum $A$ in $D$ (cf. [14, Theorem 3.1]).

**Lemma 4.** *Let $n \geqslant 2$, $p \in (n-1, n]$, $x_0 \in \mathbb{R}^n$, $\varepsilon_1 > 0$, and $A^*$ be a (non-degenerate) continuum in $D \subset \mathbb{R}^n$. Let $r > 0$ and $C_j$, $j = 1, 2, \ldots$, be a sequence of continua in $B(x_0, \varepsilon_1)$ such that $h(C_j) \geqslant r$, $h(C_j) = \sup_{x,y \in C_j} h(x, y)$. Then there is an $R_0^* > 0$ such that $M_p(\Gamma(C_j, A^*, D)) \geqslant R_0^*$ for all $j \in \mathbb{N}$.*

*Proof.* Due to Lemma 1, for any continuum $A \subset B(x_0, \varepsilon_1)$ there exists $R_0 > 0$ such that $M_p(\Gamma(C_j, A, D)) \geqslant R_0$ for all $j \in \mathbb{N}$. Thus, we may consider that $A^* \subset \mathbb{R}^n \setminus B(x_0, \varepsilon_1)$. In particular, $A \cap A^* = \varnothing$. Choose $r_0 > 0$ such that $0 < 4r_0 < \min\{r, d(A, A^*)\}$. Let $A_1, \ldots, A_q$ be a finite covering of $A$ by closed balls centered at the points $a_i \in A_i$ and of the radius $r_0$, $i = 1, \ldots, q$. Denote $\delta_i := M_p(\Gamma(A_i, A^*, D))$. Recall that $p$-modulus of families of paths joining two continua in $D$ is positive for $n-1 < p \leqslant n$ (see Proposition 3). Now, we set $R_0^* = 3^{-n} \min\left\{R_0/q, \delta_1, \ldots, \delta_q, c_n \log 2\right\}$ for $p = n$, where $c_n$ is a constant from Lemma 3, and

$$R_0^* = 3^{-p} \min\left\{R_0/q, \delta_1, \ldots, \delta_q, \frac{2^n b_{n,p}}{n-p}((2r_0)^{n-p} - r_0^{n-p})\right\}$$

for $p \neq n$, where $b_{n,p}$ is a constant from Proposition 2. Fix $j \in \mathbb{N}$. Due to the subadditivity of the modulus of families of paths, we get

$$0 < R_0 \leqslant M_p(\Gamma(A_i, C_j, D)) \leqslant M_p\left(\Gamma\left(\bigcup_{i=1}^{q} A_i, C_j, D\right)\right) \leqslant \sum_{i=1}^{q} M_p(\Gamma(A_i, C_j, D))\,. \tag{6}$$

It follows from (6) that $M_p(\Gamma(A_{i_0}, C_j, D)) \geqslant \delta/q$ at least for some $i_0 \in \{1, \ldots, q\}$. Since $A^* \cap B(a_i, 2r_0) = \varnothing$ and since $d(C_j) > 4r_0$, the assertion follows from Lemma 3 setting $F_1 = A^*$, $F_2 = C_j$ and $F_3 = A_i$. □

Given a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ and a point $x_0 \in \mathbb{R}^n$ we set

$$q_{x_0}(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S(x_0,t)} Q(x)\, d\mathcal{H}^{n-1}\,,$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure. The following lemma is proved in [23, Lemma 2.1].

**Lemma 5.** *Let $1 \leqslant p \leqslant n$, and $\Phi : [0,\infty] \to [0,\infty]$ be a strictly increasing convex function such that the relation $\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{p-1}}} = \infty$ holds for some $\delta_0 > \tau_0 := \Phi(0)$. Let $\mathfrak{Q}$ be a family of functions $Q : \mathbb{R}^n \to [0,\infty]$ such that $\int_D \Phi(Q(x))\frac{dm(x)}{(1+|x|^2)^n} \leqslant M_0 < \infty$ for some $0 < M_0 < \infty$. Now, for any $0 < r_0 < 1$ and for every $\sigma > 0$ there exists $0 < r_* = r_*(\sigma, r_0, \Phi) < r_0$ such that $\int\limits_{\varepsilon}^{r_0} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)} \geqslant \sigma$, $\varepsilon \in (0, r_*)$, for any $Q \in \mathfrak{Q}$.*

Let $D \subset \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ be a discrete open mapping, $\beta : [a,\, b) \to \mathbb{R}^n$ be a path, and $x \in f^{-1}(\beta(a))$. A path $\alpha : [a,\, c) \to D$ is called a *maximal $f$-lifting* of $\beta$ starting at $x$, if

*(a)* $\alpha(a) = x$,

*(b)* $f \circ \alpha = \beta|_{[a,\,c)}$,

*(c)* for $c < c' \leqslant b$, there is no path $\alpha' : [a,\, c') \to D$ such that $\alpha = \alpha'|_{[a,\,c)}$ and $f \circ \alpha' = \beta|_{[a,\,c')}$.

If $\beta : [a,b) \to \overline{\mathbb{R}^n}$ is a path and if $C \subset \overline{\mathbb{R}^n}$, we say that $\beta \to C$ as $t \to b$, if the spherical distance $h(\beta(t), C)$ tends to zero as $t \to b$ (see [10, Section 3.11]), where $h(\beta(t), C) = \inf\limits_{x\in C} h(\beta(t), x)$. The following assertion holds (see [10, Lemma 3.12]).

**Proposition 4.** *Let $f : D \to \mathbb{R}^n$, $n \geqslant 2$, be an open discrete mapping and $x_0 \in D$. Let $\beta : [a,\, b) \to \mathbb{R}^n$ be a path such that $\beta(a) = f(x_0)$ and such that either $\lim\limits_{t\to b}\beta(t)$ exists, or $\beta(t) \to \partial f(D)$ as $t \to b$. Then $\beta$ has a maximal $f$-lifting $\alpha : [a,\, c) \to D$ starting at $x_0$. If $\alpha(t) \to x_1 \in D$ as $t \to c$, then $c = b$ and $f(x_1) = \lim\limits_{t\to b}\beta(t)$. Otherwise $\alpha(t) \to \partial D$ as $t \to c$.*

*Proof of Theorem 1.* We proceed by contradiction. Assume that its conclusion is wrong, i.e. there is a compactum $K$ in $D$ for which there exists $0 < \varepsilon_1 < \operatorname{dist}(K, \partial D)$ such that for any $m \in \mathbb{N}$ there exists $f_m \in \mathfrak{F}^{Q,p}_{a,b,\delta}(D)$ and $x_m \in K$ such that $B_h\big(f_m(x_m), \frac{1}{m}\big) \setminus f_m(B(x_m, \varepsilon_1)) \neq \varnothing$, where $B_h\big(f_m(x_m), \frac{1}{m}\big) = \{w \in \overline{\mathbb{R}^n} : h(w, f_m(x_m)) < \frac{1}{m}\}$. Let $y_m \in B_h(f_m(x_m), \frac{1}{m}) \setminus f_m(B(x_m, \varepsilon_0))$. Due to the compactness of $\overline{\mathbb{R}^n}$ we may consider that $y_m \to y_0$ as $m \to \infty$, where $y_0 \in \overline{\mathbb{R}^n}$. Then also $f_m(x_m) \to y_0$ as $m \to \infty$. Let us firstly consider that $y_0 \neq \infty$. Passing to a subsequence, if necessary, we may consider that $|f_m(x_m) - y_m| < 1/m$, $m = 1, 2, \ldots$.

Let us join the points $a$ and $b$ with a path $\gamma : [0,1] \to D$, $\gamma(0) = a$, $\gamma(1) = b$, lying in $D$. It follows that $h(f_m(\gamma)) \geqslant \delta$ for any $m = 1, 2, \ldots$, $f_m \in \mathfrak{F}^{Q,p}_{a,b,\delta}(D)$. We may consider that the sequences $f_m(a)$ and $f_m(b)$ converge to some points $z_1$ and $z_2$ as $m \to \infty$ because $\overline{\mathbb{R}^n}$ is a compact space. Due to the condition $h(f_m(a), f_m(b)) \geqslant \delta$, at least one of the above points does not coincide with $y_0$. Without loss of generality, let $z_1 \neq y_0$. Since $f_m$ is equicontinuous at $a$, given $\sigma > 0$ there is $\chi = \chi(\sigma)$ such that $h(f_m(x), f_m(a)) < \sigma$ for $|x - a| < \chi$. We can chose numbers $r_1, r_2 > 0$ small enough so that

$$B_h(z_1, r_1) \cap B(y_0, r_2) = \varnothing\,. \tag{7}$$

By the triangle inequality, $h(f_m(x), z_1) \leqslant h(f_m(x), f_m(a)) + h(f_m(a), z_1) < \sigma + h(f_m(a), z_1)$ for $|x - a| < \chi$. Since $h(f_m(a), z_1) \to 0$ as $m \to \infty$ the latter relation implies that $f_m(x) \in B_h(z_1, r_1)$ for sufficiently large $m$ and $\sigma = r_1/2$. Let $E = \{|x - a| < \chi\}$, where $\chi$ is mentioned above.

Join the points $y_m$ and $f_m(x_0)$ by the segment $\beta_m : [0,1] \to \overline{B(f_m(x_m), 1/m)}$ such that $\beta_m(0) = f_m(x_m)$ and $\beta_m(0) = y_m$. Let $\alpha_m$, $\alpha_m : [0, c_m) \to B(x_m, \varepsilon_1)$, be a maximal $f_m$-lifting of $\beta_m$ in $B(x_m, \varepsilon_1)$ starting at $x_0$, which exists by Proposition 4. By the same proposition either

$\alpha_m(t) \to x_1 \in B(x_m, \varepsilon_1)$ as $t \to c_m - 0$ (in this case, $c_m = 1$ and $f_m(x_1) = y_m$), or $\alpha_m(t) \to S(x_m, \varepsilon_1)$ as $t \to c_m$. Observe that the first situation is excluded. Indeed, if $f_m(x_1) = y_m$, then $y_m \in f_m(B(x_m, \varepsilon_1))$, that contradicts the choice of $y_m$. Thus, $\alpha_m(t) \to S(x_m, \varepsilon_1)$ as $t \to c_m$. Observe that, $\overline{|\alpha_m|}$ is a continuum in $\overline{B(x_m, \varepsilon_1)}$ and $h(\overline{|\alpha_m|}) \geqslant h(x_m, S(x_m, \varepsilon_1))$.

Without loss of generality we may consider that $x_m \to x_0 \in K$, $m \to \infty$. Let us show that $\overline{|\alpha_m|} \in B(x_0, \varepsilon_1^*)$, where $\varepsilon_1^* > 0$ is some number with $0 < \varepsilon_1^* < \operatorname{dist}(K, \partial D)$. Indeed, let $x \in \overline{|\alpha_m|}$. Since $\overline{|\alpha_m|}$ is a continuum in $\overline{B(x_m, \varepsilon_1)}$, by the triangle inequality we obtain that $|x - x_0| \leqslant |x - x_m| + |x_m - x_0| < \varepsilon_1 + |x_m - x_0|$. Since $x_m - x_0 \to 0$ as $m \to \infty$ and since $0 < \varepsilon_1 < \operatorname{dist}(K, \partial D)$, we may chose $r_* > 0$ small enough so that $\varepsilon_1^* := \varepsilon_1 + r_* < \operatorname{dist}(K, \partial D)$. Thus, $\overline{|\alpha_m|} \in B(x_0, \varepsilon_1^*)$, $0 < \varepsilon_1^* < \operatorname{dist}(K, \partial D)$, as required.

Recall that $h(\overline{|\alpha_m|}) \geqslant h(x_m, S(x_m, \varepsilon_1))$. Let $h(x_m, S(x_m, \varepsilon_1)) = h(x_m, w_m)$, where $w_m \in S(x_m, \varepsilon_1)$. Now, by the definition of a chordal metrics, we have

$$h(w_m, x_0) = \frac{|w_m - x_0|}{\sqrt{1 + |w_m|^2}\sqrt{1 + |x_0|^2}}. \tag{8}$$

By the triangle inequality $|w_m| \leqslant |w_m - x_0| + |x_0| \leqslant |w_m - x_m| + |x_m - x_0| + |x_0| \leqslant 2\varepsilon_1 + |x_0|$ for sufficiently large $m$, we obtain from (8) that

$$h(x_m, w_m) = \frac{|w_m - x_m|}{\sqrt{1 + |w_m|^2}\sqrt{1 + |x_0|^2}} \geqslant \frac{\varepsilon_1}{\sqrt{1 + |2\varepsilon_1 + |x_0||^2}\sqrt{1 + |x_0|^2}} := r\,. \tag{9}$$

Thus, we get $h(\overline{|\alpha_m|}) \geqslant r$ for sufficiently large $m \in \mathbb{N}$. Now, let us apply Lemma 4 for $A := E = \{|x - x_0| < \chi\}$, $C_m := |\alpha_m|$ and $r$, which is defined in (9). By this lemma we may find $R_0 > 0$ such that

$$M_p(\Gamma(\overline{|\alpha_m|}, E, D)) \geqslant R_0\,, \qquad m = 1, 2, \ldots\,. \tag{10}$$

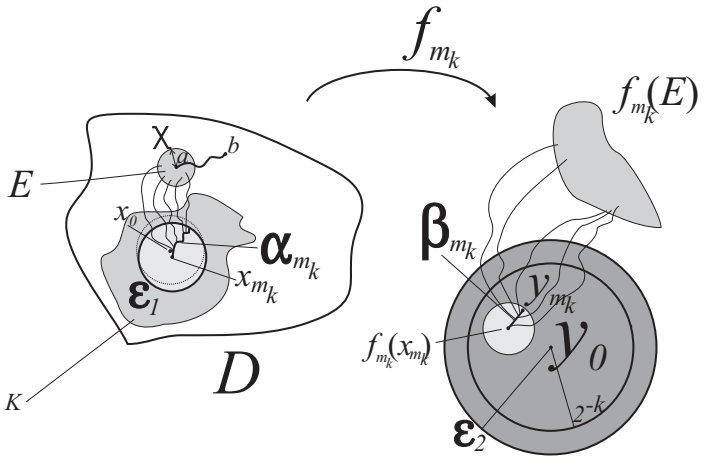


**Figure 1.** To the proof of Theorem 1

Let us show that the relation (10) contradicts the definition of the mapping $f_m$ in (1)–(2). Indeed, since $f_m(x_m) \to y_0$ as $m \to \infty$, for any $k \in \mathbb{N}$ there is a number $m_k \in \mathbb{N}$ such that

$$B(f_{m_k}(x_{m_k}), 1/m_k) \subset B(y_0, 2^{-k})\,. \tag{11}$$

Since $|\beta_m| \in B(f_m(x_m), 1/m)$, by (11) we obtain that

$$|\beta_{m_k}| \subset B(y_0, 2^{-k}), \qquad k = 1, 2, \ldots . \tag{12}$$

Let $k_0 \in \mathbb{N}$ be such that $2^{-k} < \varepsilon_2$, where $\varepsilon_2 < r_2$ and $r_2$ corresponds to (7), and let $\Gamma_k := \Gamma(|\alpha_{m_k}|, E, D)$. In this case, $\Gamma_k > \Gamma_{f_{m_k}}(y_0, 2^{-k}, \varepsilon_2)$ for sufficiently large $k \in \mathbb{N}$, see (7) and (12) (cf. Figure 1).

Now, by the definition of $f_{m_k}$ in (1)–(2), we obtain that

$$\begin{aligned} M_p(\Gamma_k) = M_p(\Gamma(|\alpha_{m_k}|, E, D)) &\leqslant M_p(\Gamma_{f_{m_k}}(y_0, 2^{-k}, \varepsilon_2)) \\ &\leqslant \int\limits_{A(y_0, 2^{-k}, \varepsilon_2)} Q_{f_{m_k}}(y) \cdot \eta^{\,p}(|y - y_0|)\, dm(y) \leqslant \int\limits_{A(y_0, 2^{-k}, \varepsilon_2)} \widetilde{Q}_{f_{m_k}}(y) \cdot \eta^{\,p}(|y - y_0|)\, dm(y)\,, \end{aligned}$$

where $\widetilde{Q}_{f_{m_k}}(y) = \max\left\{Q_{f_{m_k}}(y), 1\right\}$, and $\eta : (2^{-k}, \varepsilon_2) \to \mathbb{R}$ is any nonnegative Lebesgue measurable function with $\int\limits_{2^{-k}}^{\varepsilon_2} \eta(r)\, dr \geqslant 1$. Now, we set $I_k = I(y_0, 2^{-k}, \varepsilon_2) = \int\limits_{2^{-k}}^{\varepsilon_2} \frac{dr}{r^{\frac{n-1}{p-1}} \widetilde{q}_{k,y_0}^{\frac{1}{p-1}}(r)}$, where

$$\widetilde{q}_{k,y_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(y_0, r)} \widetilde{Q}_{f_{m_k}}(y)\, d\mathcal{H}^{n-1}\,.$$

By Lemma 5 and Remark 1, we get

$$I_k = \int\limits_{2^{-k}}^{\varepsilon_2} \frac{dr}{r^{\frac{n-1}{p-1}} \widetilde{q}_{k,y_0}^{\frac{1}{p-1}}(r)} \to \infty \tag{13}$$

as $k \to \infty$. Since $\widetilde{q}_{k,y_0}(r) \geqslant 1$ for a.e. $r$ and by (13), $0 < I_k < \infty$ for sufficiently large $k \in \mathbb{N}$. Set

$$\psi_k(t) = \begin{cases} 1/[t^{\frac{n-1}{p-1}} \widetilde{q}_{k,y_0}^{\frac{1}{p-1}}(t)], & t \in (2^{-k}, \varepsilon_2), \\ 0, & t \notin (2^{-k}, \varepsilon_2). \end{cases}$$

Let $\eta_k(t) = \psi_k(t)/I_k$ for $t \in (2^{-k}, \varepsilon_2)$ and $\eta_k(t) = 0$ otherwise. Now, $\eta_k$ satisfies (2) for $r_1 = 2^{-k}$ and $r_2 = \varepsilon_2$. Therefore, by (3), (13) and by Fubini's theorem, we obtain

$$M_p(\Gamma_k) \leqslant \frac{1}{I_k^p} \int\limits_{A(y_0, 2^{-k}, \varepsilon_2)} Q(y) \cdot \psi_k^p(|y - y_0|)\, dm(y) = \frac{\omega_{n-1}}{I_k^{p-1}} \to 0 \tag{14}$$

as $k \to \infty$. The relation (14) contradicts with (10).

Let now $y_0 = \infty$. Now, we set $\widetilde{f}_m := \psi \circ f_m$ and $\widetilde{y}_m := \psi(y_m)$, where $\psi(y) = \frac{y}{|y|^2}$. Since $y_0 = \infty$, we obtain that $\widetilde{y}_m \to 0$ as $m \to \infty$. Besides that, since $f_m(x_m) - y_m \to 0$ as $m \to \infty$, by the triangle inequality, $|\widetilde{f}_m(x_m) - \widetilde{y}_m| = |\psi(f_m(x_m)) - \psi(y_m)| \leqslant |\psi(f_m(x_m))| + |\psi(y_m)| \to 0$ as $m \to \infty$. Therefore, $\widetilde{f}_m(x_m) - \widetilde{y}_m \to 0$ as $m \to \infty$. Passing to a subsequence, we may also consider that $|\widetilde{f}_m(x_m) - \widetilde{y}_m| < \frac{1}{m}$, $m = 1, 2, \ldots$. In addition, by the assumption of the theorem, the mappings $\widetilde{f}_m$ satisfy the relations (1)–(2) at the origin with a new function $\widetilde{Q}(y) := Q\left(\frac{y}{|y_2|}\right)$. The conformal change of the variables $z = \frac{y}{|y|^2}$ corresponds to the Jacobian $J(z, y) = \frac{1}{|y|^{2n}}$. So, observe that

$$\int\limits_{\mathbb{R}^n} \Phi\left(Q_{f_m}\left(\frac{y}{|y|^2}\right)\right) \cdot \frac{dm(y)}{(1 + |y|^2)^n} = \int\limits_{\mathbb{R}^n} \Phi(Q_{f_m}(z)) \cdot \frac{dm(z)}{(1 + |z|^2)^n} < \infty\,. \tag{15}$$

The relation (15) implies the possibility of applying above technique to $Q_{f_m}\left(\frac{y}{|y|^2}\right)$ instead of $Q_{f_m}(y)$. In particular, the relation (15) together with (4) implies by Lemma 5 that (13) holds for $\widetilde{\widetilde{q}}_{k,y_0}(r) := \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{S(y_0,r)} \widetilde{Q}_{f_{m_k}}\left(\frac{y}{|y|^2}\right) d\mathcal{H}^{n-1}$ instead of $\widetilde{q}_{k,y_0}$. Finally, since the family $\{f_m\}_{m=1}^{\infty}$ is equicontinuous at $a$ and $b$, the family $\{\widetilde{f}_m\}_{m=1}^{\infty}$ is also equicontinuous at these points because $\widetilde{f}_m = \psi \circ f_m$ and $\psi$ is a fixed continuous function.

Now, taking into account what has been said, we repeat verbatim the proof of Theorem 1 for the case $y_0 = 0$ and for $\widetilde{f}_m$ instead $f_m$, $m = 1, 2, \ldots$. Repeating this proof, we obtain the desired conclusion. □

The following result is taken from [16, Proposition 4.4.I].

**Proposition 5.** *Let $f$ and $g$ be homotopic via a homotopy $h_t$, $t \in [0,1]$, $h_0 = f$, $h_1 = g$. Suppose further that $y$ is $(h_t, U)$-admissible for all $t \in [0,1]$. Then $\mu(y, f, U) = \mu(y, g, U)$, where $\mu(y, f, G)$ denotes the topological degree of $f$ at $y$ with respect to $G$.*

Given a map $f : D \to \mathbb{R}^n$, a set $E \subset D$ and $y \in \mathbb{R}^n$, we define the *multiplicity function* $N(y, f, E)$ as a number of preimages of the point $y$ in a set $E$, i.e.

$$N(y, f, E) = \text{card } \{x \in E : f(x) = y\}, \qquad N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E). \tag{16}$$

The following result is taken from [16, Proposition 4.10.I].

**Proposition 6.** *The relation $N(y, f, G) \leqslant \mu(y, f, G)$ is true for $y \in \mathbb{R}^n \setminus f(\partial G)$ and any compact subdomain $G$ of $D$.*

*Proof of Theorem 2.* Assume that $f$ is not a constant. Let us to show that $f$ is open. Let $A$ be an open set and let $x_0 \in A$. We need to show that there is $\varepsilon^* > 0$ such that $B(f(x_0), \varepsilon^*) \subset f(A)$. Since $A$ is open, there is $\varepsilon_1 > 0$ such that $\overline{B(x_0, \varepsilon_1)} \subset A$. Since $f$ is not constant, there are $a, b \in D$ such that $f(a) \neq f(b)$. Let us join the points $a$ and $b$ by a path $\gamma$ in $D$. We set $E := |\gamma|$. Now, $h(f_m(a), f_m(b)) \geqslant \frac{1}{2} \cdot h(f(a), f(b)) := \delta$ for sufficiently large $m \in \mathbb{N}$. By Theorem 1 there is $r_0 > 0$, which does not depend on $m$, such that $B_h(f_m(x_0), r_0) \subset f_m(B(x_0, \varepsilon_1))$, $m = 1, 2, \ldots$.

Set $\varepsilon^* := \frac{r_0}{2}$. Let $y \in B_h(f(x_0), r_0/2)$. Since by assumption $f_m(x) \to f(x)$ locally uniformly in $D$, by the triangle inequality we get $h(f_m(x_0), y) \leqslant h(f_m(x_0), f(x_0)) + h(f(x_0), y) < r_0$ for sufficiently large $m \in \mathbb{N}$. Thus, $y \in B_h(f_m(x_0), r_0) \subset f_m(B(x_0, \varepsilon_1))$. Consequently, $y = f_m(x_m)$ for some $x_m \in B(x_0, \varepsilon_1)$. Due to the compactness of $\overline{B(x_0, \varepsilon_1)}$, we may consider that $x_m \to z_0 \in \overline{B(x_0, \varepsilon_1)}$ as $m \to \infty$. By the continuity of $f$ in $A$, since $\overline{B(x_0, \varepsilon_1)} \subset A$, we obtain $f(x_m) \to f(z_0)$ as $m \to \infty$. So, we have $f(x_m) \to f(z_0)$ as $m \to \infty$ and simultaneously $y = f_m(x_m)$ for large enough $m \in \mathbb{N}$. Therefore

$$|y - f(z_0)| = |f_m(x_m) - f(z_0)| \leqslant |f_m(x_m) - f(x_m)| + |f(x_m) - f(z_0)| \to 0 \quad \text{as} \quad m \to \infty.$$

Thus, $y = f(z_0) \in f(\overline{B(x_0, \varepsilon_1)}) \subset f(A)$. So, $y \in f(A)$, i.e. $B(f(x_0), r_0/2) \subset f(A)$, as required.

We will prove the discreteness of the mapping using methodology from [5]. Assume the contrary. Then there is $x_0 \in D$ and a sequence $x_k \in D$, $k = 1, 2, \ldots$, $x_k \neq x_0$, such that $x_k \to x_0$ as $k \to \infty$ and $f(x_k) = f(x_0)$. Observe that $E_0 = \{x \in D : f(x) = f(x_0)\}$ is closed in $D$ by the

continuity of $f$ and does not coincide with $D$, because $f \not\equiv const$. Thus, we may assume that $x_0$ may be replaced by non isolated boundary point of $E_0$. Let $\varepsilon_0 > 0$ be such that $\overline{B(x_0, \varepsilon_0)} \subset D$. By the assumption, $f$ is not identically a constant in $B(x_0, \varepsilon)$ for any $0 < \varepsilon < \varepsilon_0$.

We show that for every $0 < \varepsilon < \varepsilon_0$ there exist $0 < t < \varepsilon_0$ such that $f(x_0) \notin f(S(x_0, t))$. Otherwise, there is $0 < \varepsilon < \varepsilon_0$ and $x_t \in S(x_0, t)$ with $f(x_t) = f(x_0)$ for every $0 < t < \varepsilon$. Since $f$ is not a constant in $B(x_0, \varepsilon)$, we may find $z_2 \in B(x_0, \varepsilon)$ and $\varepsilon_2 > 0$ such that $\overline{B(z_2, \varepsilon_2)} \subset B(x_0, \varepsilon)$ and $\overline{B(f(x_0), \varepsilon_2)} \cap f(B(z_2, \varepsilon_2)) = \varnothing$. Since $f_m$ converges to $f$ locally uniformly, we also have

$$d(f_m(B(z_2, \varepsilon_2)), f(x_0)) \geqslant \varepsilon_1 > 0 \tag{17}$$

for large enough $m \in \mathbb{N}$ and some $\varepsilon_1 > 0$.

Let $E = \{x_t\}_{t \in (0, \varepsilon)}$, $\Gamma = \Gamma(\overline{B(z_2, \varepsilon_2)}, E, B(x_0, \varepsilon))$. Since $\overline{B(z_2, \varepsilon_2)}$ is a continuum, there are $0 < a < b < \varepsilon$ such that $E \cap S(x_0, t) \neq \varnothing \neq \overline{B(z_2, \varepsilon_2)} \cap S(x_0, t)$ for every $a < t < b$. By Proposition 2, we get

$$M_p(\Gamma) \geqslant \alpha > 0 \tag{18}$$

for some $\alpha > 0$. On the other hand, let $F := f^{-1}(f(x_0)) \cap \overline{B(x_0, \varepsilon)}$. Now, $F$ is a compact set in $D$. We set $r_m = \max\{0, \widetilde{r}_m\}$, where $\widetilde{r}_m = \sup_{x \in f_m(E)} |x - f(x_0)|$. By the triangle inequality, $|f_m(x) - f_m(x_0)| \leqslant |f_m(x) - f(x)| + |f(x) - f(x_0)| \to 0$ as $m \to \infty$ uniformly over $x \in F$. Thus, $\widetilde{r}_m = \sup_{x \in f_m(E)} |x - f(x_0)| \leqslant \sup_{x \in f_m(F)} |x - f(x_0)| = |f_m(z_m) - f(x_0)| \to 0$ as $m \to \infty$, where $z_m \in F$ is such that $\sup_{x \in f_m(F)} |x - f(x_0)| = |f_m(z_m) - f(x_0)|$. Therefore, $f_m(E) \subset B(f(x_0), r_m)$, where $r_m \to 0$ as $m \to \infty$. Now, by (17), $f_m(\Gamma) > \Gamma(S(f(x_0), r_m), S(f(x_0), \varepsilon_1), A(f(x_0), r_m, \varepsilon_1))$ for sufficiently large $m \in \mathbb{N}$. The latter implies that $\Gamma > \Gamma_{f_m}(f(x_0), r_m, \varepsilon_1)$. This inequality together with the definition of $f_{m_k}$ in (1)–(2) gives that

$$M_p(\Gamma) \leqslant M_p(\Gamma_{f_m}(f(x_0), r_m, \varepsilon_1)) \leqslant \int\limits_{A(f(x_0), r_m, \varepsilon_1)} \widetilde{Q}_{f_m}(y) \cdot \eta^{\,p}(|y - f(x_0)|)\, dm(y)\,, \tag{19}$$

where $\widetilde{Q}_{f_m}(y) = \max\left\{Q_{f_m}(y), 1\right\}$, whenever $\eta : (r_m, \varepsilon_1) \to \mathbb{R}$ is an arbitrary nonnegative Lebesgue measurable function with $\int\limits_{r_m}^{\varepsilon_2} \eta(r)\, dr \geqslant 1$. Set $I_m = I(f(x_0), r_m, \varepsilon_1) = \int\limits_{r_m}^{\varepsilon_1} \frac{dr}{r^{\frac{n-1}{p-1}} \widetilde{q}^{\frac{1}{p-1}}_{m, f(x_0)}(r)}$, where $\widetilde{q}_{m, f(x_0)}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(f(x_0), r)} \widetilde{Q}_{f_m}(y)\, d\mathcal{H}^{n-1}$. By Lemma 5 and Remark 1, we get

$$I_m = \int\limits_{r_m}^{\varepsilon_1} \frac{dr}{r^{\frac{n-1}{p-1}} \widetilde{q}^{\frac{1}{p-1}}_{m, f(x_0)}(r)} \to \infty \quad \text{as} \quad m \to \infty. \tag{20}$$

Since $\widetilde{q}_{m, f(x_0)}(r) \geqslant 1$ for a.e. $r$ and by (20), $0 < I_m < \infty$ for sufficiently large $m \in \mathbb{N}$. Set

$$\psi_m(t) = \begin{cases} 1/[t^{\frac{n-1}{p-1}} \widetilde{q}^{\frac{1}{p-1}}_{m, f(x_0)}(t)], & t \in (r_m, \varepsilon_1), \\ 0, & t \notin (r_m, \varepsilon_1). \end{cases}$$

Let $\eta_m(t) = \psi_m(t)/I_m$ for $t \in (r_m, \varepsilon_1)$ and $\eta_m(t) = 0$ otherwise. Therefore, $\eta_m$ satisfies (2) for $r_1 = r_m$ and $r_2 = \varepsilon_1$. Now, by (19), (20) and by Fubini's theorem, we obtain

$$M_p(\Gamma) \leqslant \frac{1}{I_m^p} \int\limits_{A(f(x_0), r_m, \varepsilon_1)} Q(y) \cdot \psi_m^p(|y - f(x_0)|)\, dm(y) = \frac{\omega_{n-1}}{I_m^{p-1}} \to 0 \quad \text{as} \quad m \to \infty.$$

The above relation contradicts with (18). Thus, for every $0 < \varepsilon < \varepsilon_0$ there exist $0 < t < \varepsilon$ such that $f(x_0) \notin f(S(x_0, t))$, as required.

Let $0 < \rho < \varepsilon_0$, $0 < \varepsilon < \rho$ be such that $y = f(x_0) \notin f(S(x_0, \varepsilon))$. Let $\beta = d(y, f(S(x_0, \varepsilon))) > 0$. By Proposition 5, $n_m := \mu(y, f_m, B(x_0, \varepsilon)) = \mu(y, f, B(x_0, \varepsilon)) = q(y)$ for large enough $m \in \mathbb{N}$. Now, by Proposition 6, we get

$$N(y, f_m, B(x_0, \varepsilon)) \leqslant n_m = q(y) < \infty \tag{21}$$

for sufficiently large $m = 1, 2, \ldots$. Since, by the assumption, $f$ is not discrete, we may find $x_1, \ldots, x_j \in f^{-1}f(x_0)$ in $B(x_0, \varepsilon)$ with $j > q = q(y)$. We can choose $\varepsilon_k > 0$ such that $\overline{B(x_k, \varepsilon_k)} \subset B(x_0, \varepsilon)$ for $k = 1, 2, \ldots$, while $\overline{B(x_k, \varepsilon_k)} \cap \overline{B(x_p, \varepsilon_p)} = \varnothing$ for $k \neq p$, $1 \leqslant k, p \leqslant j$.

By Theorem 1, there is $r_k > 0$, which does not depend on $m$, such that

$$B_h(f_m(x_k), r_k) \subset f_m(B(x_k, \varepsilon_k)), \quad k = 1, 2, \ldots j, \quad m = 1, 2, \ldots .$$

The latter implies that, for $m \geqslant m_0$, some $m_0 \in \mathbb{N}$ and $r_\varepsilon > 0$ we have

$$B_h(f(x_0), r_\varepsilon) \subset f_m(B(x_k, \varepsilon_k)), \quad k = 1, 2, \ldots j, \quad m \geqslant m_0 . \tag{22}$$

The relation (22) implies that there are $a_k \in B(x_k, \varepsilon_k)$ such that $f_m(a_k) = f(x_0) = y$, $k = 1, 2, \ldots, j$. The latter means that $N(y, f_m, B(x_0, \varepsilon)) \geqslant j > q$ that contradicts with (21). The obtained contradiction proves the discreteness of $f$, that finishes the proof. □

The following statement holds (cf. [5] and [22]).

**Theorem 3.** *Let $D$ be a domain in $\mathbb{R}^n$, $n \geqslant 2$. Let $n - 1 < p \leqslant n$ and $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ be a non-decreasing function. Besides that, let $f_j : D \to \mathbb{R}^n$, $n \geqslant 2$, $j = 1, 2, \ldots$, be a sequence of open discrete mappings satisfying the conditions* (1)–(2) *with some function $Q = Q_j : \mathbb{R}^n \to [0, \infty]$ at any point $y_0 \in \overline{\mathbb{R}^n}$ for which* (3) *holds with $Q_{f_j} := Q_j$ and converging to some mapping $f : D \to \overline{\mathbb{R}^n}$ as $j \to \infty$ locally uniformly in $D$ with respect to the chordal metric $h$. Assume that the condition* (4) *holds. Then either $f$ is a constant in $\overline{\mathbb{R}^n}$, or $f$ is open discrete map $f : D \to \mathbb{R}^n$.*

*Proof.* The proof is based on Theorem 1 and completely similar to the first part of the proof of [23, Theorem 1.2]. □

## 4 Connection with Orlicz-Sobolev classes

The definitions of Sobolev and Orlicz-Sobolev classes used below can be found in [24] and are therefore omitted. We also do not provide many of the notations used below.

If $\rho : \mathbb{R}^n \to [0, \infty]$ is a Borel function, the integral of $\rho$ over $S$ is defined as

$$\int\limits_S \rho^k \, d\mathcal{A} = \int\limits_{\mathbb{R}^n} \rho^k(y) N(S, y) \, d\mathcal{H}^k y. \tag{23}$$

Let $\Gamma$ be a family of $k$-dimensional surfaces $S$. A Borel function $\rho\colon \mathbb{R}^n \to \overline{\mathbb{R}^+}$ is said to be *admissible* for $\Gamma$ (briefly $\rho \in \mathrm{adm}\Gamma$) if

$$\int\limits_S \rho^k \, d\mathcal{A} \geqslant 1 \tag{24}$$

for every surface $S \in \Gamma$, where the integral on the left-hand side of (24) is defined by relation (23). We say that a property $P$ holds for almost every $k$-dimensional surface, if $P$ holds for all surfaces except a family of zero $p$-modulus.

The following important information concerning the capacity of a pair of sets relative to a domain can be found in the paper by W.P. Ziemer [26]. Let $G$ be a bounded domain in $\mathbb{R}^n$, and let $C_0, C_1$ be nonintersecting compact subsets of the closure of $G$. Put $R = G \setminus (C_0 \cup C_1)$ and $R^* = R \cup C_0 \cup C_1$, then the *p-capacity of the pair $C_0, C_1$ relative to the closure of $G$* is defined to be the quantity $C_p[G, C_0, C_1] = \inf \int_R |\nabla u|^p \, dm(x)$, where the infimum is taken over all functions $u$ continuous on $R^*$, $u \in ACL(R)$, with $u = 1$ on $C_1$ and $u = 0$ on $C_0$. Such functions will be called admissible for $C_p[G, C_0, C_1]$. A set $\sigma \subset \mathbb{R}^n$ is said to *separate $C_0$ and $C_1$ in $R^*$* if $\sigma \cap R$ is closed in $R$ and there exist nonintersecting sets $A$ and $B$ open in $R^* \setminus \sigma$ and such that $R^* \setminus \sigma = A \cup B$, $C_0 \subset A$, and $C_1 \subset B$. Let $\Sigma$ denote the class of all sets separating $C_0$ and $C_1$ in $R^*$. Putting $p' = p/(p-1)$, we introduce the quantity $\widetilde{M}_{p'}(\Sigma) = \inf_{\rho \in \widetilde{\mathrm{adm}\,\Sigma}} \int_{\mathbb{R}^n} \rho^{p'} \, dm(x)$, where the formula $\rho \in \widetilde{\mathrm{adm}\,\Sigma}$ means that $\rho$ is a nonnegative Borel function on $\mathbb{R}^n$ such that

$$\int_{\sigma \cap R} \rho \, d\mathcal{H}^{n-1} \geqslant 1 \qquad \forall\, \sigma \in \Sigma. \tag{25}$$

Observe that, by Ziemer's result, we have

$$\widetilde{M}_{p'}(\Sigma) = C_p[G, C_0, C_1]^{\,-1/(p-1)}, \tag{26}$$

see [26, Theorem 3.13] for $p = n$ and [27, p. 50] for $1 < p < \infty$. We also observe that, by a result of J. Hesse, we get

$$M_p(\Gamma(E, F, D)) = C_p[D, E, F]\,, \tag{27}$$

see [8, Theorem 5.5].

The following class of mappings is a generalization of quasiconformal mappings in the sense of the ring definition by Gehring (see, e.g., [11, Chapter 9]). Let $p \geqslant 1$, let $D$ and $D'$ be domains in $\overline{\mathbb{R}^n}$, $n \geqslant 2$, $x_0 \in \mathbb{R}^n$ and let $Q : D \to (0, \infty)$ be a Lebesgue measurable function. We say that $f : D \to D'$ is a *lower Q-mapping at $x_0$ with respect to the p-modulus,* whenever

$$M_p(f(\Sigma_\varepsilon)) \geqslant \inf_{\rho \in \mathrm{ext}_p \,\mathrm{adm}\,\Sigma_\varepsilon} \int_{D \cap A(x_0, \varepsilon, r_0)} \frac{\rho^p(x)}{Q(x)} \, dm(x) \tag{28}$$

for each ring $A(x_0, \varepsilon, r_0) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < r_0\}$, $r_0 \in (0, d_0)$, $d_0 = \sup_{x \in D} |x - x_0|$, where $\Sigma_\varepsilon$ denotes the family of all intersections of spheres $S(x_0, r)$ with domain $D$, $r \in (\varepsilon, r_0)$. The following statement facilitates the verification of the infinite series of inequalities in (28) and may be established similarly to the proof of [11, Theorem 9.2].

**Lemma 6.** *Let $D, D' \subset \overline{\mathbb{R}^n}$, $x_0 \in \overline{D} \setminus \{\infty\}$ and let $Q : D \to (0, \infty)$ be a Lebesgue measurable function. Then a mapping $f : D \to D'$ is a lower Q-mapping with respect to the p-modulus at $x_0$, $p > n - 1$, if and only if*

$$M_p(f(\Sigma_\varepsilon)) \geqslant \int_\varepsilon^{r_0} \frac{dr}{\|Q\|_s(r)} \quad \forall\, \varepsilon \in (0, r_0),\ r_0 \in (0, d_0),\ d_0 = \sup_{x \in D} |x - x_0|,\ s = \frac{n-1}{p-n+1},$$

*where, as above, $\Sigma_\varepsilon$ denotes the family of all intersections of spheres $S(x_0, r)$ with the domain $D$, $r \in (\varepsilon, r_0)$, $\|Q\|_s(r) = \Big( \int\limits_{D(x_0, r)} Q^s(x) \, d\mathcal{A} \Big)^{\frac{1}{s}}$ is an $L_s$-norm of function $Q$ over sphere $D(x_0, r) = \{x \in D : |x - x_0| = r\} = D \cap S(x_0, r)$.*

For any $x \in D$ and fixed $p, q$, $p, q \geqslant 1$, let us define

$$K_{I,q}(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^q}, & J(x,f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{29}$$

The following statement was first proved for homeomorphisms and $x_0 \in \overline{D}$ in [9, Theorem 2.1] (see also [19, Lemma 2.3]). The proof of this statement for the case $x_0 \in \mathbb{R}^n$ does not differ from the above case $x_0 \in \overline{D}$.

**Lemma 7.** *Let $D$ be a domain in $\mathbb{R}^n$, $n \geqslant 3$, $\varphi : (0,\infty) \to (0,\infty)$ be a non-decreasing function satisfying the condition $\int_1^\infty \left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}} dt < \infty$. If $n \geqslant 3$ and $p > n-1$, then every open discrete mapping $f : D \to \mathbb{R}^n$ with finite distortion of the class $W^{1,\varphi}_{loc}$ such that $N(f,D) < \infty$ is a lower $Q$-mapping with respect to $p$-modulus at each point $x_0 \in \mathbb{R}^n$ for $Q(x) = N(f,D) \cdot K_{I,\alpha}^{\frac{p-n+1}{n-1}}(x,f)$, $\alpha := \frac{p}{p-n+1}$, where the inner dilatation $K_{I,\alpha}(x,f)$ of $f$ at $x$ is of order $\alpha$ is defined by the relation (29), and the multiplicity $N(f,D)$ is defined by the second relation in* (16).

In what follows we will need the following auxiliary assertion (see, e.g., [11, Lemma 7.4, Ch. 7] for $p = n$ and [18, Lemma 2] for $p \neq n$.

**Proposition 7.** *Let $x_0 \in \mathbb{R}^n$, $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function, $Q \in L^1_{loc}(\mathbb{R}^n)$. We set $A := A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}$ and $\eta_0(r) = \frac{1}{Ir^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}$, where*

$$I := I = I(x_0, r_1, r_2) = \int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)} \qquad \text{and} \qquad q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{|x-x_0|=r} Q(x)\, d\mathcal{H}^{n-1}$$

*is the integral average of the function $Q$ over the sphere $S(x_0, r)$. Then*

$$\frac{\omega_{n-1}}{I^{p-1}} = \int\limits_A Q(x) \cdot \eta_0^p(|x - x_0|)\, dm(x) \leqslant \int\limits_A Q(x) \cdot \eta^p(|x - x_0|)\, dm(x)$$

*for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0,\infty]$ such that $\int\limits_{r_1}^{r_2} \eta(r) dr = 1$.*

The following statement was proved for some cases earlier (see, e.g., [9, Proposition 3]).

**Theorem 4.** *Let $x_0 \in \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ be a bounded lower $Q$-homeomorphism with respect to $p$-modulus in a domain $D \subset \mathbb{R}^n$, $Q \in L^{\frac{n-1}{p-n+1}}_{loc}(\mathbb{R}^n)$, $n-1 < p \leqslant n$, $\alpha := \frac{p}{p-n+1}$. Then for every $0 < \varepsilon < \varepsilon_0 < \infty$ and any compact sets $C_2 \subset D \setminus B(x_0, \varepsilon_0)$ and $C_1 \subset \overline{B(x_0,\varepsilon)} \cap D$ the inequality*

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leqslant \int\limits_{A(x_0,\varepsilon,\varepsilon_1)} Q^{\frac{n-1}{p-n+1}}(x) \eta^\alpha(|x - x_0|) dm(x)$$

*holds, where $A(x_0, \varepsilon, \varepsilon_1) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_1\}$ and $\eta : (\varepsilon, \varepsilon_1) \to [0,\infty]$ is an arbitrary Lebesgue measurable function such that $\int\limits_\varepsilon^{\varepsilon_1} \eta(r) dr = 1$.*

*Proof.* Note that the set $\sigma_r := f(S(x_0, r) \cap D)$ is closed in $f(D)$. In addition, note that $\sigma_r$ separates $f(C_1)$ from $f(C_2)$ for $r \in (\varepsilon, \varepsilon_0)$ in $f(D)$, since

$$f(C_1) \subset f(B(x_0, r) \cap D) := A, \qquad f(C_2) \subset f(D) \setminus \overline{f(B(x_0, r) \cap D)} := B,$$

$A$ and $B$ are open in $f(D)$ and $f(D) = A \cup \sigma_r \cup B$. Let $\Sigma_\varepsilon$ be the family of all sets separating $f(C_1)$ from $f(C_2)$ in $f(D)$. Let $\rho^{n-1} \in \widetilde{\operatorname{adm}} \bigcup_{r\in(\varepsilon,\varepsilon_0)} f(S(x_0, r) \cap D)$ in the sense of the relation (25). Then also $\rho \in \operatorname{adm} \bigcup_{r\in(\varepsilon,\varepsilon_0)} f(S(x_0, r) \cap D)$ in the sense of the relation (24) for $k = n - 1$. Therefore, since $\widetilde{M}_q(\Sigma_\varepsilon) \geqslant M_{q(n-1)}(\Sigma_\varepsilon)$ for arbitrary $q \geqslant 1$, we have that

$$\widetilde{M}_{p/(n-1)}(\Sigma_\varepsilon) \geqslant M_p\left( \bigcup_{r\in(\varepsilon,\varepsilon_1)} f(S(x_0, r) \cap D)\right). \tag{30}$$

However, due to (26) and (27), $\widetilde{M}_{p/(n-1)}(\Sigma_\varepsilon) = \frac{1}{(M_\alpha(\Gamma(f(C_1),f(C_2),f(D))))^{1/(\alpha-1)}}$. By Lemma 6,

$$M_p\left( \bigcup_{r\in(\varepsilon,\varepsilon_1)} f(S(x_0, r) \cap D)\right) \geqslant \int_\varepsilon^{\varepsilon_1} \frac{dr}{\| Q\|_s(r)} = \int_\varepsilon^{\varepsilon_1} \frac{dt}{\omega_{n-1}^{\frac{p-n+1}{n-1}} t^{\frac{n-1}{\alpha-1}} \widetilde{q}_{x_0}^{\frac{1}{\alpha-1}}(t)} \tag{31}$$

for all $n \in \mathbb{N}$, where $s = \frac{n-1}{p-n+1}$, $\|Q\|_s(r) = \left( \int_{D(x_0,r)} Q^s(x)\, d\mathcal{A}\right)^{\frac{1}{s}}$ is $L_s$-norm of the function $Q$ over the sphere $S(x_0, r) \cap D$ and $\widetilde{q}_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x-x_0|=r} Q^s(x)\, d\mathcal{H}^{n-1}$. Then from (30)–(31) it follows that

$$M_\alpha(\Gamma(f(C_1), f(C_2), f(D))) \leqslant \frac{\omega_{n-1}}{I^{\alpha-1}}, \tag{32}$$

where $I = \int\limits_\varepsilon^{\varepsilon_1} \frac{dr}{r^{\frac{n-1}{\alpha-1}} \widetilde{q}_{x_0}^{\frac{1}{\alpha-1}}(r)}$. Note that $f(\Gamma(C_1, C_2, D)) \subset \Gamma(f(C_1), f(C_2), f(D))$, so that from (32) it follows that $M_\alpha(f(\Gamma(C_1, C_2, D))) \leqslant \frac{\omega_{n-1}}{I^{\alpha-1}}$. This completes the proof by Proposition 7. □

The following result follows from Lemma 7 and Theorem 4.

**Theorem 5.** *Let $D$ be a domain in $\mathbb{R}^n$, $n \geqslant 3$, $\varphi : (0, \infty) \to (0, \infty)$ be a non-decreasing function satisfying condition*

$$\int_1^\infty \left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}} dt < \infty. \tag{33}$$

*If $0 < \varepsilon < \varepsilon_0 < \infty$, $n \geqslant 3$ and $n - 1 < \alpha \leqslant n$, then every bounded homeomorphism $f : D \to \mathbb{R}^n$ with finite distortion in $W^{1,\varphi}_{loc}$ with $K_{I,\alpha}(x, f) \in L^1_{\rm loc}(D)$ satisfies the relation*

$$M_\alpha(f(\Gamma(C_1, C_2, D))) \leqslant \int_{A(x_0,\varepsilon,\varepsilon_1)} Q(x)\eta^\alpha(|x - x_0|)\, dm(x)$$

*for $Q(x) = K_{I,\alpha}(x, f)$, $A(x_0, \varepsilon, \varepsilon_1) = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_1\}$, for any compact sets $C_2 \subset D \setminus B(x_0, \varepsilon_0)$ and $C_1 \subset \overline{B(x_0, \varepsilon)} \cap D$ and for an arbitrary Lebesgue measurable function $\eta : (\varepsilon, \varepsilon_1) \to [0, \infty]$ such that $\int\limits_\varepsilon^{\varepsilon_1} \eta(r)dr = 1$.*

Let now $f : D \to D'$, $x_0 \in D$ and $y_0 = f(x_0)$. Assume that $f$ is a homeomorphism and denote $g := f^{-1}$. Observe that $f(\Gamma(C_1, C_2, D)) = f(\Gamma(S(x_0, \varepsilon)), S(x_0, \varepsilon_0), D)$ and, consequently, $\Gamma(C_1, C_2, D) = \Gamma_g(x_0, \varepsilon, \varepsilon_0)$.

Given $p \geqslant 1$, a non-decreasing function $\Phi\colon \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$, $a, b \in D$, $a \neq b$, $\delta > 0$, we denote by $\mathfrak{O}^{\Phi,p}_{a,b,\delta}(D)$ the family of all homeomorphisms $f : D \to \mathbb{R}^n$, $n \geqslant 2$, such that its inverses $g : f(D) \to D$ are of the class $W^{1,\varphi}_{loc}$ and have finite distortion such that

$$\int\limits_{\mathbb{R}^n} \Phi(K_{I,p}(y,g))\frac{dm(y)}{(1+|y|^2)^n} < \infty$$

and $h(f(a), f(b)) \geqslant \delta$.

**Theorem 6.** *Let $n \geqslant 2$, $p \in (n-1, n]$, and $D$ be a bounded domain in $\mathbb{R}^n$. Assume that the corresponding family $\mathfrak{O}^{\Phi,p}_{a,b,\delta}(D)$ is equicontinuous at $a$ and $b$. In addition, $\Phi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is an increasing convex function that satisfies the condition*

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau(\Phi^{-1}(\tau))^{\frac{1}{p-1}}} = \infty$$

*for some $\delta > \Phi(0)$. Assume that the relation* (33) *holds.*

*Then for every compactum $K$ in $D$ and for every $0 < \varepsilon < \operatorname{dist}(K, \partial D)$ there exists $r_0 = r_0(\varepsilon, K) > 0$ which does not depend on $f$ such that $f(B(x_0, \varepsilon)) \supset B_h(f(x_0), r_0)$ for all $f \in \mathfrak{O}^{\Phi,p,\varphi}_{a,b,\delta}(D)$ and all $x_0 \in K$, where $B_h(f(x_0), r_0) = \{w \in \overline{\mathbb{R}^n} : h(w, f(x_0)) < r_0\}$.*

*Proof.* By Theorem 5 and comments made before the formulation of Theorem 6, we obtain $\mathfrak{O}^{\Phi,p,\varphi}_{a,b,\delta}(D) \subset \mathfrak{F}^{\Phi,p}_{a,b,\delta}(D)$. The rest follows by Theorem 1. □

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Ількевич Н.С., Севостьянов Є.О., Таргонський В.А. *Про теорему Кебе для відображень з інтегральними обмеженнями* // Карпатські матем. публ. — 2026. — Т.18, №2. — С. 487–503.

Досліджено відображення, що задовольняють обернену модульну нерівність типу Полецького відносно порядка $p$. За умови $n-1 < p \leqslant n$ доведено, що образ деякої кулі містить кулю фіксованого радіуса рівномірно по вказаній вище сім'ї відображень. Це твердження можна інтерпретувати як аналог відомої теореми Кебе для аналітичних функцій, отриманий в більш загальних класах відображень. Як наслідок, отримано відкритість та дискретність граничного відображення в класі, що досліджується. У статті також окремо вивчаються класи Орліча-Соболєва, для яких отримано аналог теореми Кебе про одну чверть.

*Ключові слова і фрази:* квазіконформне відображення, відображення зі скінченним спотворенням, відображення класів Орліча-Соболєва, теорема Кебе.